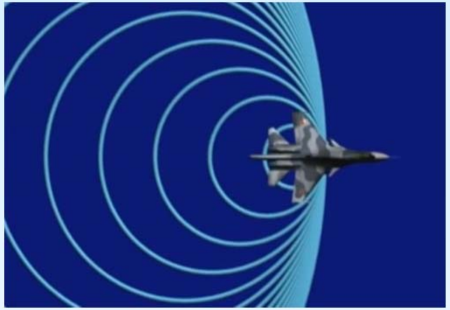
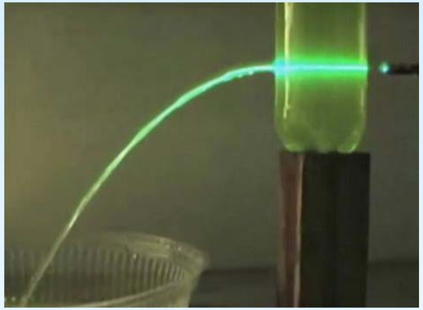
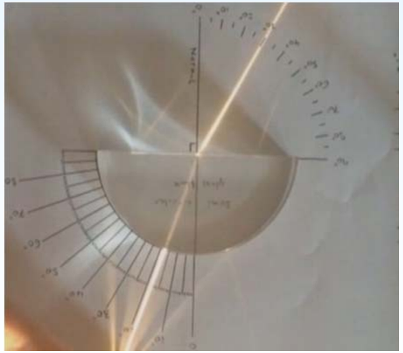
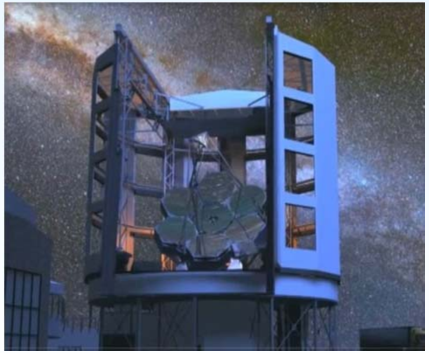
Wavemechanics and optics





Chapter 14 – Harmonic oscillation

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- Part 2. Problems
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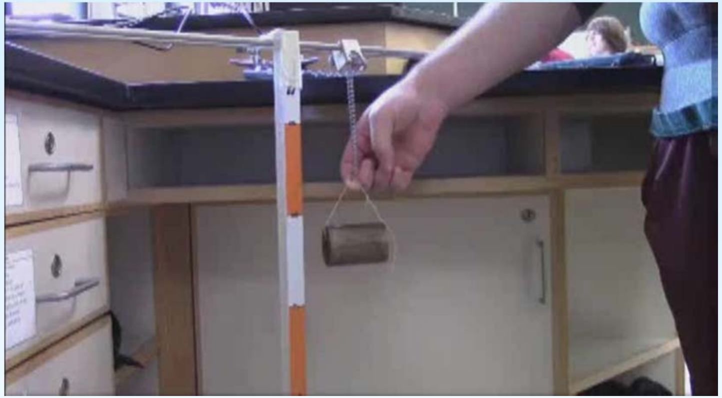
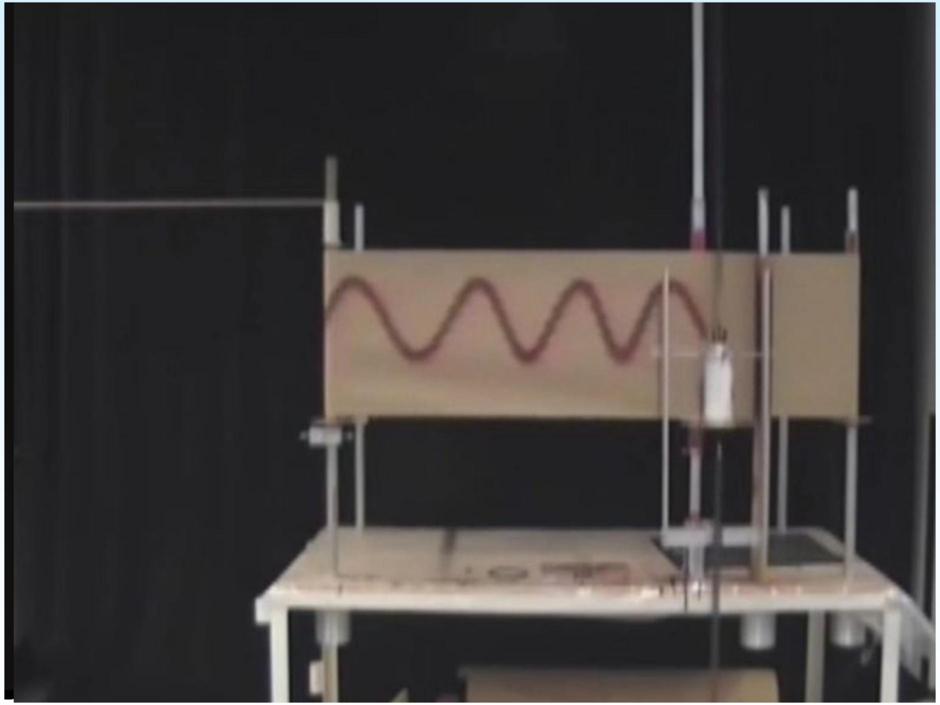
Part 1. What is harmonic oscillation and how can we describe it mathematically ?

Harmonic oscillation: Examples



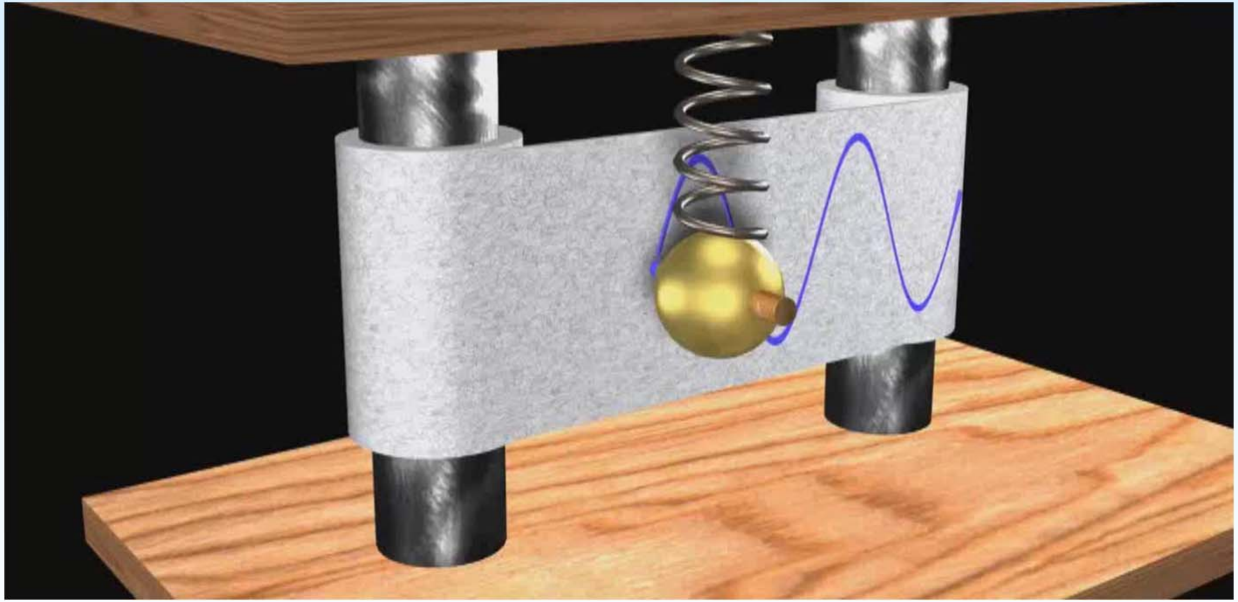
Harmonic oscillation: Experiment

An experiment to find a mathematical description of harmonic oscillation:



https://www.youtube.com/watch?v=p9uhmjbZn-c

Harmonic oscillation: Experiment

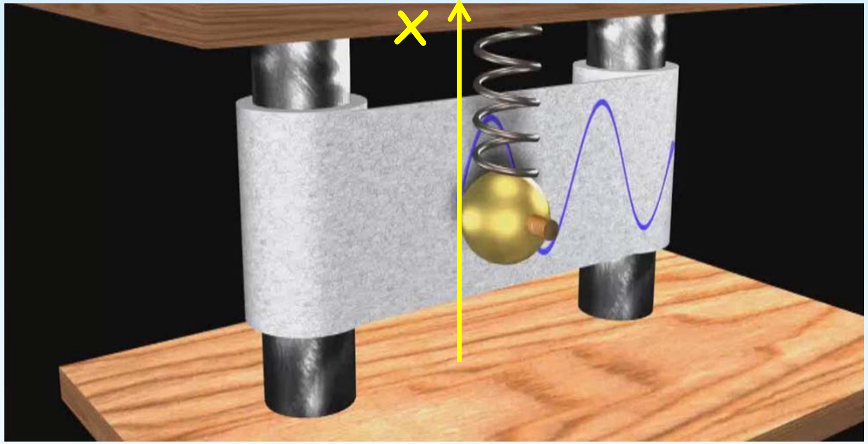


Conclusion: Harmonic oscillation can be described by the function:

$$x = A \sin(Bt + C)$$

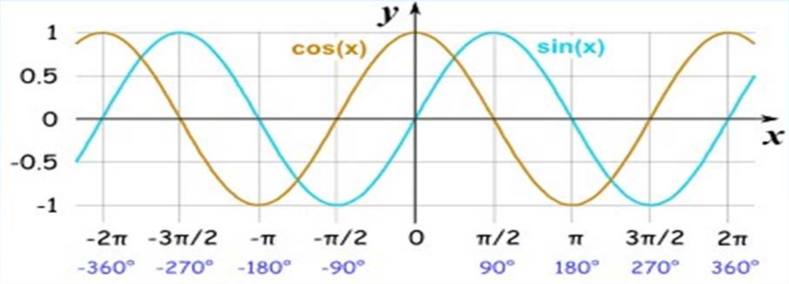
if t is the time and A, B and C are constants that describes the motion.

Harmonic oscillation: Notation



$x = A \sin(Bt + C)$ or

$x = A \cos(Bt + C - \pi/2)$



x : Vertical displacement. Unit: meter

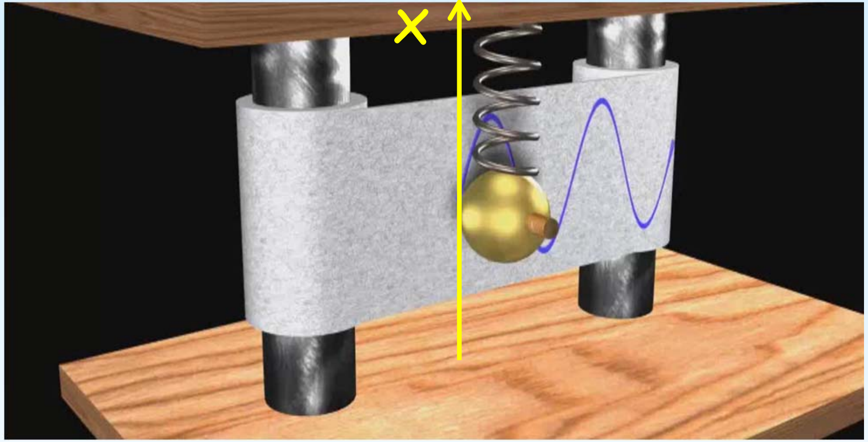
t : Time. Unit: second

A : Amplitude (maximum displacement). Unit: meter

B = ω : Angular frequency (the number of oscillations per second times 2π).
Unit: Radians per second

C = ϕ : Phase angle (determines the position at time = 0). Unit: radians

Harmonic oscillation: Notation



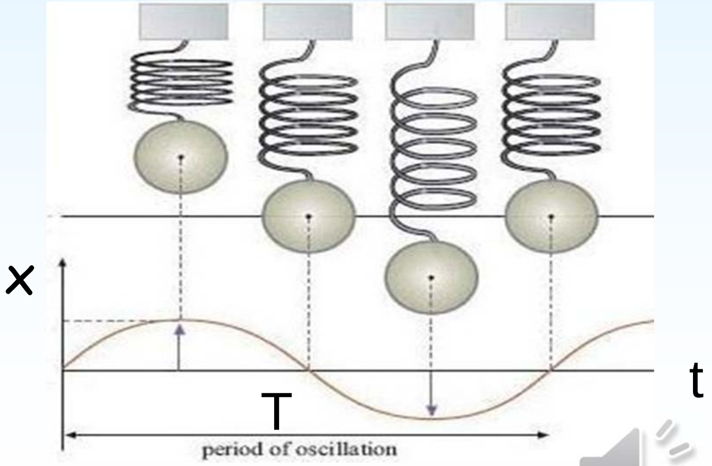
$$x = A\sin(\omega t + \phi')$$

or

$$x = A\cos(\omega t + \phi)$$

T: Period = the time it takes for the weight to go up and down. Unit: second

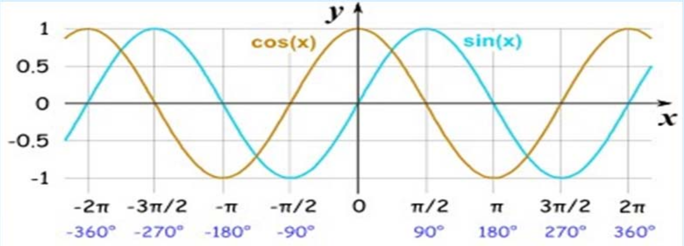
f: Frequency = the number of periods per second. Unit: 1/second = Hz



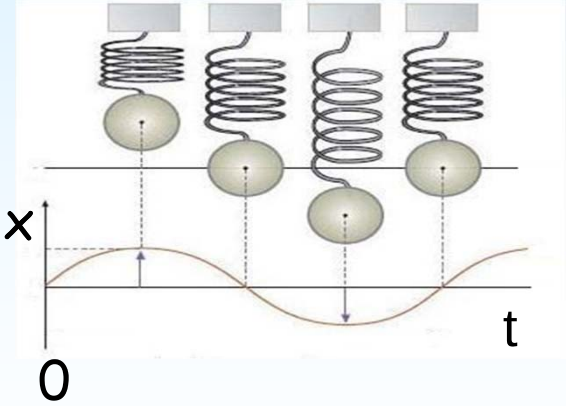
$f = 1 / T$ $\omega = 2\pi f$

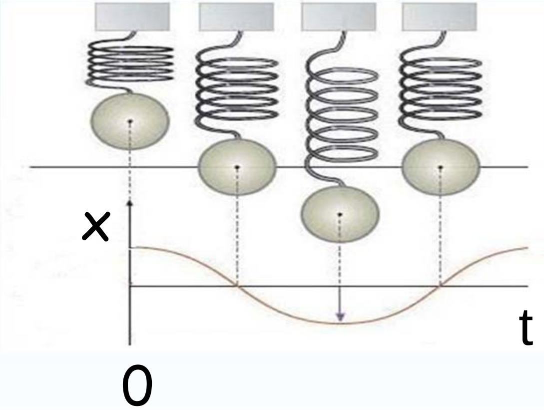
Harmonic oscillation: Phase angle

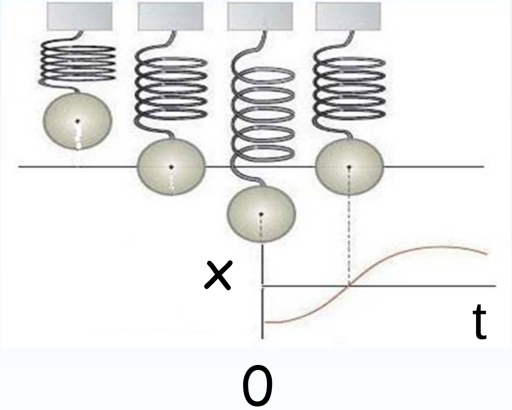
$x = A\sin(\omega t + \phi')$ or $x = A\cos(\omega t + \phi)$



The phase angle (ϕ) determines the position at time = 0 since then $x = A\sin(\phi')$ or $x = A\cos(\phi)$







$X = A\sin(\omega t)$

$X = A\cos(\omega t - \pi/2)$

$X = A\cos(\omega t)$

$X = A\sin(\omega t + \pi/2)$

$X = A\cos(\omega t + \pi)$

$X = A\sin(\omega t - \pi/2)$

We now have a mathematical description of the displacement.

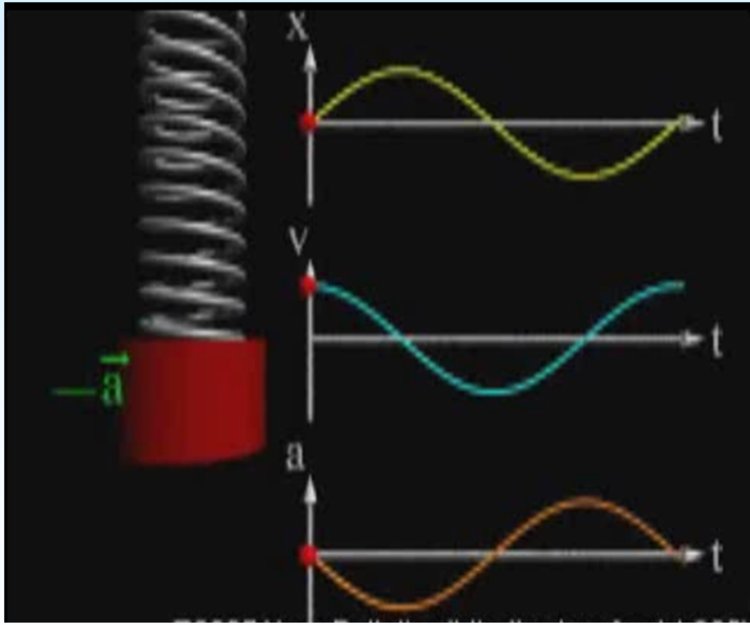
What is the velocity and acceleration ?

$$v(t) = \frac{dx}{dt}$$

$$a(t) = \frac{dv}{dt}$$

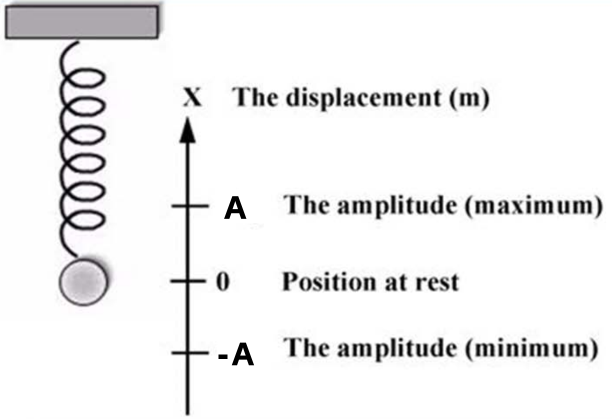
Harmonic oscillation: velocity & acceleration

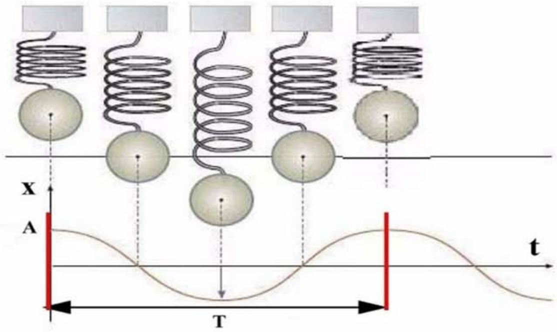
Displacement:	$x = A\sin(\omega t)$	$\longrightarrow$	$x_{max} = A$
Velocity:	$v = \omega A\cos(\omega t)$	$\longrightarrow$	$v_{max} = \omega A$
Acceleration:	$a = -\omega^2 A\sin(\omega t)$	$\longrightarrow$	$a_{max} = \omega^2 A$



https://www.youtube.com/watch?v=eeYRkW8V7Vg

Harmonic oscillation: Summary





$\phi = \text{acos}(x_0 / A) = \text{acos}(A/A) = 0$

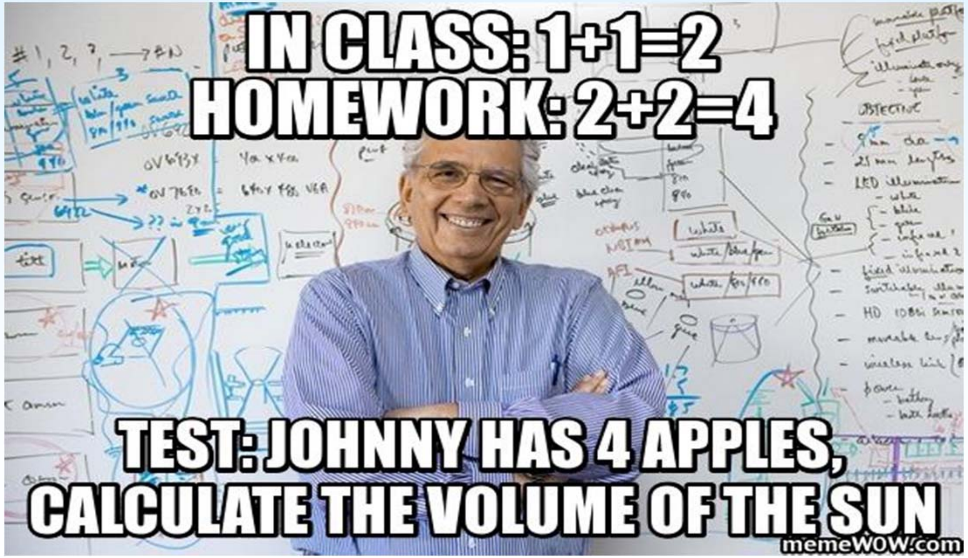
- **x** The displacement (m)
- **A** The amplitude (m)
- **t** Time (s)
- **T** Period (s)
- **f** Frequency (Hz) = 1 / T
- **ω** Angular Frequency (Hz) = 2π / T = 2 π f

$$x = A\cos(\omega t + \phi) \longrightarrow x_{max} = A$$

$$v = \frac{dx}{dt} = -\omega A\sin(\omega t + \phi) \longrightarrow v_{max} = \omega A$$

$$a = \frac{dv}{dt} = -\omega^2 A\cos(\omega t + \phi) \longrightarrow a_{max} = \omega^2 A$$

Part 2. Problems



Harmonic oscillation: Problems

An ultrasonic device uses sound at a frequency of 6.7×10^6 Hz.

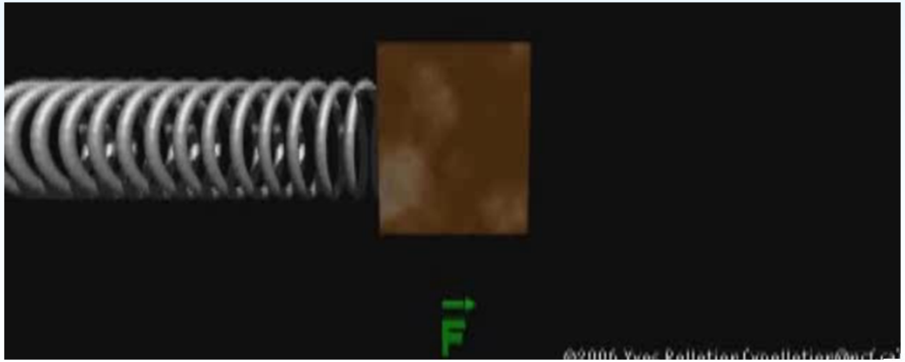
How long does each oscillation take and what angular frequency does this correspond to ?

$$f = 1/T$$
$$\omega = 2\pi f$$

$$T = \frac{1}{f} = \frac{1}{6.7 \times 10^6 \text{ Hz}} = 1.5 \times 10^{-7} \text{ s} = 0.15\ \mu\text{s}$$

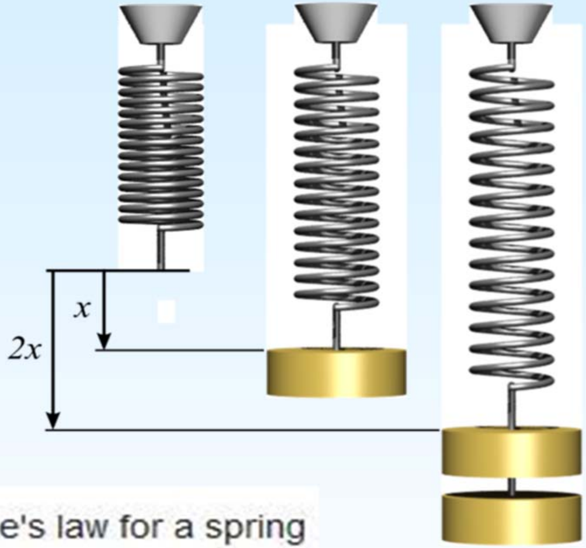
$$\begin{aligned} \omega &= 2\pi f = 2\pi(6.7 \times 10^6 \text{ Hz}) \\ &= (2\pi \text{ rad/cycle})(6.7 \times 10^6 \text{ cycle/s}) \\ &= 4.2 \times 10^7 \text{ rad/s} \end{aligned}$$

Part 3. Springs, Hooke's law & Forces



https://www.youtube.com/watch?v=_ca77OYbeZw

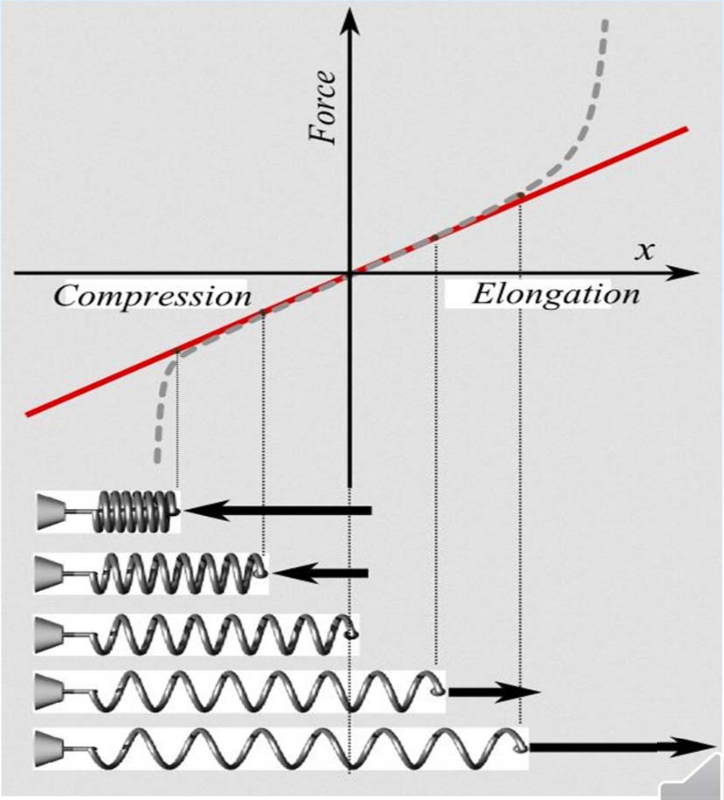
Harmonic oscillation: The spring



Hooke's law for a spring

$$F = -kX$$

k = spring constant
which describes how stiff the spring is.



Harmonic oscillation: Forces

Newton's first law of motion: A body acted on by no net force moves with constant velocity (which may be zero) and zero acceleration.

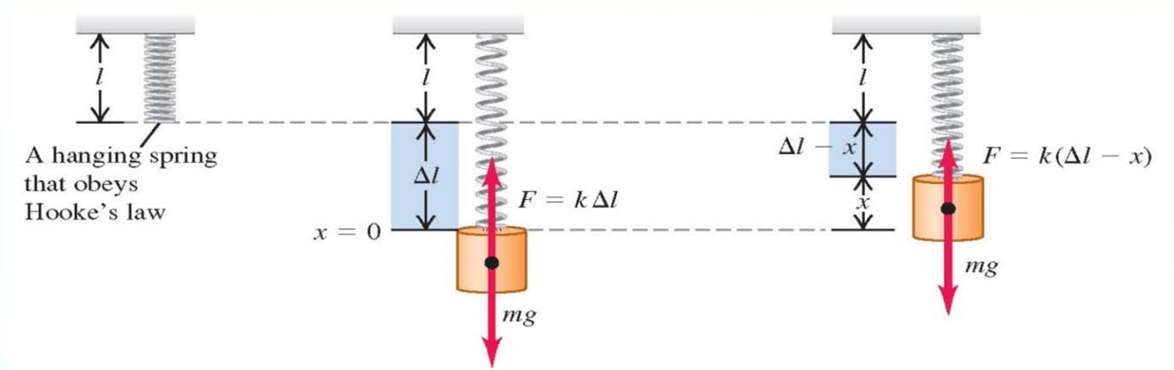
Newton's second law of motion: If a net external force acts on a body, the body accelerates. The direction of acceleration is the same as the direction of the net force. The mass of the body times the acceleration of the body equals the net force vector.

$$\sum \vec{F} = m\vec{a} \qquad \text{(Newton's second law of motion)}$$

Harmonic oscillation with a spring

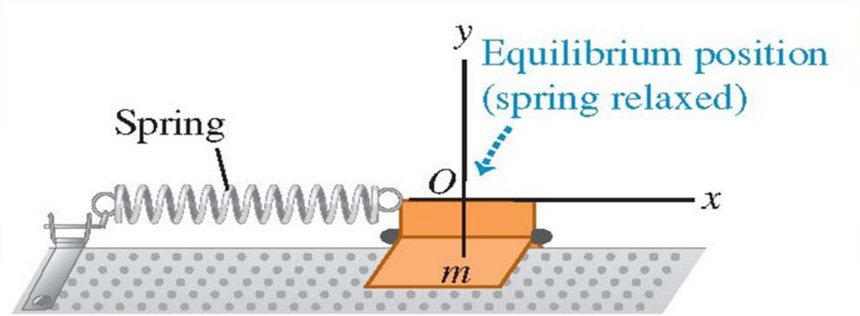
Vertical oscillation

Gravity will stretch the spring to a new eqilibrium position.



Horizontal oscillation

Gravity will not stretch the spring to a new eqilibrium position.



However, the oscillations will be the same.

Harmonic oscillation with a spring

Horizontal oscillation on an airbed.

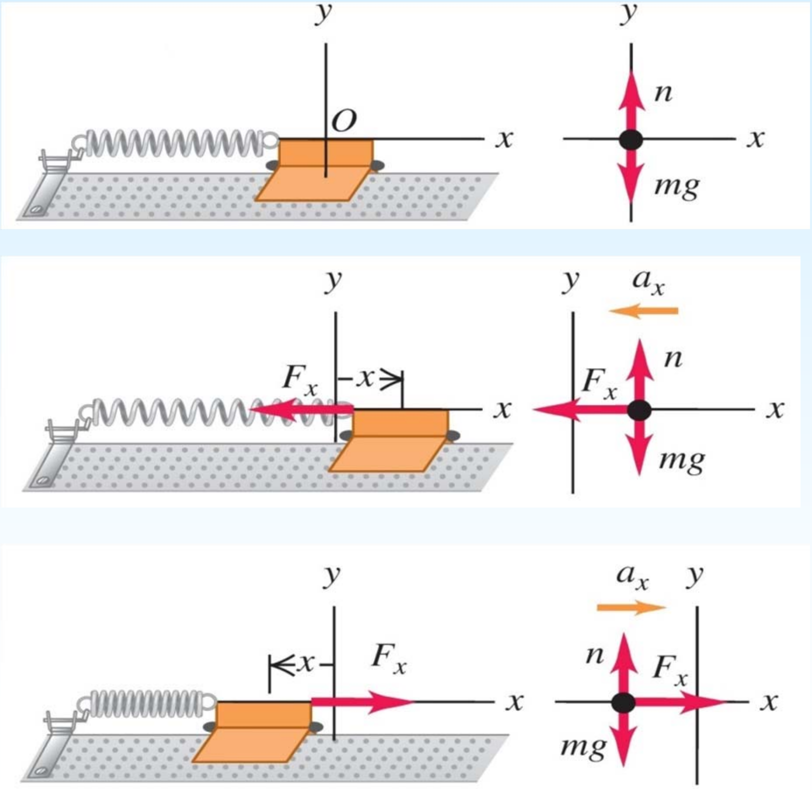
https://www.youtube.com/watch?v=9nLedU7qvvw

Harmonic oscillation: Forces

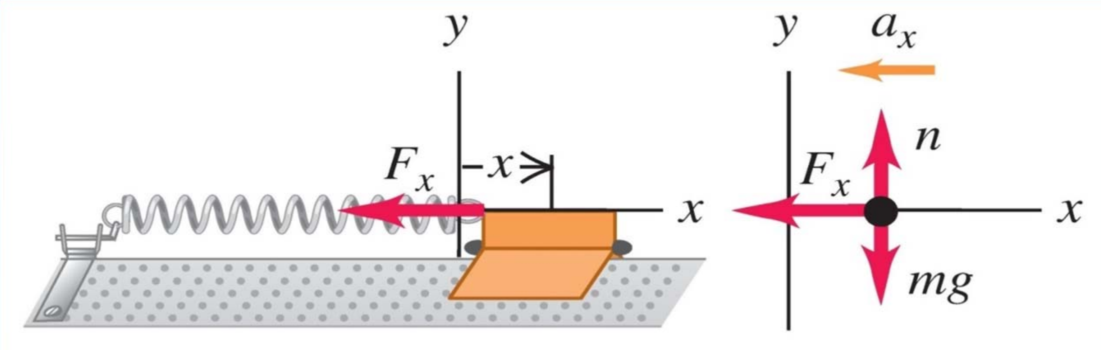
$x = 0 \quad F_{total} = 0 \quad a_x = 0$

$x > 0 \quad F_{total} < 0 \quad a_x < 0$

$x < 0 \quad F_{total} > 0 \quad a_x > 0$



Harmonic oscillation: Forces



$F_x = -kx$ (restoring force exerted by an ideal spring)

$\sum \vec{F} = m\vec{a}$ (Newton's second law of motion)

$$a_x = \frac{d^2x}{dt^2} = -\frac{k}{m}x$$ (simple harmonic motion)

Harmonic oscillation: Forces

Old formulas:

$x = A\cos(\omega t+\phi)$

$v = -\omega A\sin(\omega t+\phi)$

$a = -\omega^2 A\cos(\omega t+\phi)$

$\Rightarrow a_x = -\omega^2 x$

New formula:

$$a_x = \frac{d^2x}{dt^2} = -\frac{k}{m}x \quad \text{(simple harmonic motion)}$$

Combine old and new:

$$-\omega^2 x = -\frac{k}{m}x$$

$$\omega = \sqrt{\frac{k}{m}}$$

The frequency depends on two variables:

1. The spring constant
2. The mass

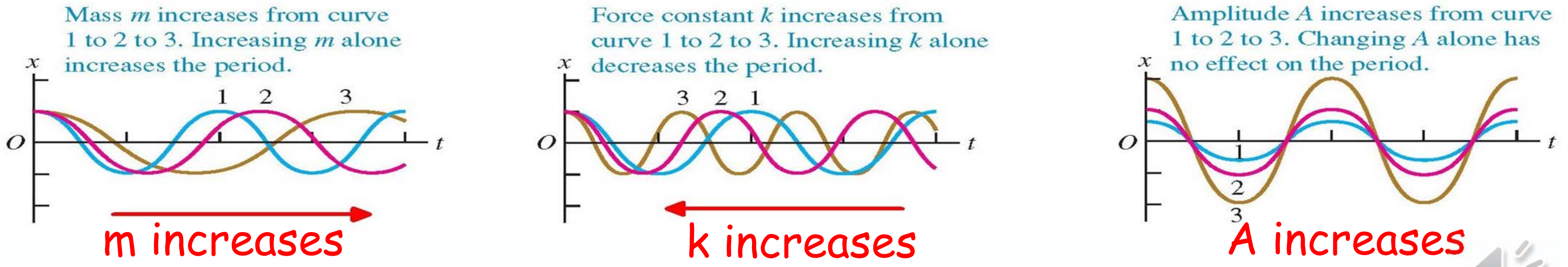
Harmonic oscillation: Forces

$$\omega = \sqrt{\frac{k}{m}}$$

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \quad \text{(simple harmonic motion)}$$

$$T = \frac{1}{f} = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} \quad \text{(simple harmonic motion)}$$

Note: f and T depend only on k and m.
Not the amplitude !



Harmonic oscillation with a spring

Increase the mass

Increase the spring constant

$$\omega = \sqrt{\frac{k}{m}}$$

The frequency decreases

The frequency increases

Harmonic oscillation: Forces

You can look at the oscillations in a different way

$$a_x = \frac{d^2x}{dt^2} = -\frac{k}{m}x \qquad \text{(simple harmonic motion)}$$

$$\frac{\partial^2 x}{\partial t^2} + \frac{k}{m}x = 0$$

This is a differential equation that has the solution

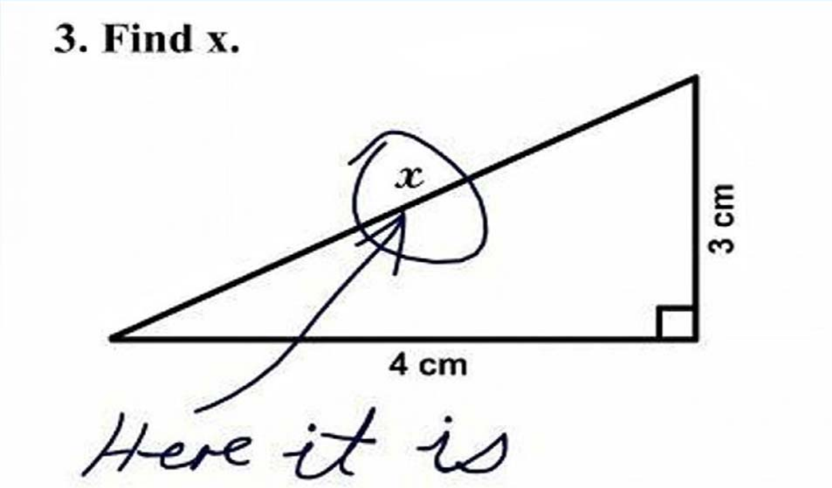
$$x = A\cos(\omega t + \varphi)$$

$$\omega = \sqrt{\frac{k}{m}}$$

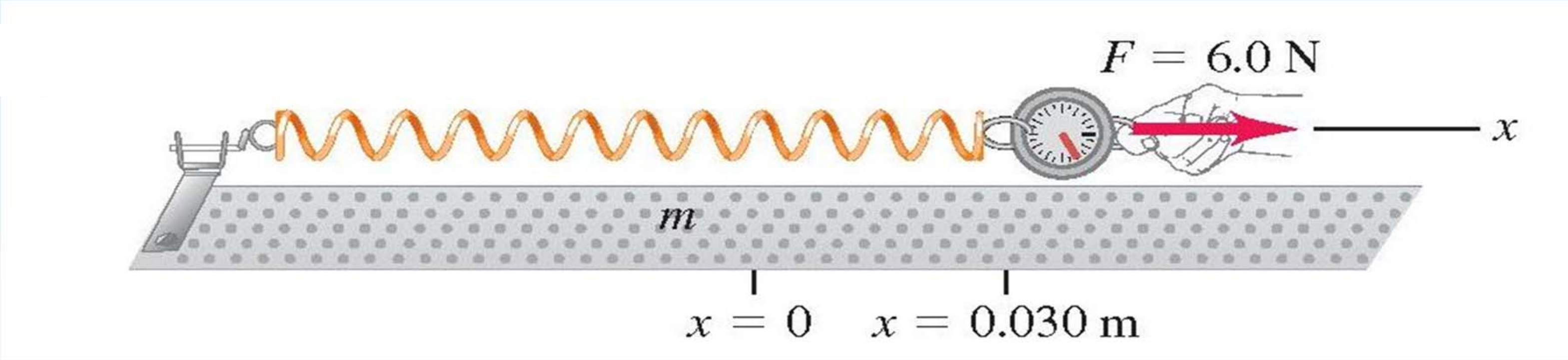
$$-\omega^2 A\cos(\omega t + \varphi) + \frac{k}{m}A\cos(\omega t + \varphi) = 0$$

$$-\omega^2 A\cos(\omega t + \varphi) + \omega^2 A\cos(\omega t + \varphi) = 0$$

Part 4. Problems



Harmonic oscillation: Problem



What is the spring constant ?

Note: The spring force is in the negative direction

F = ma

N = kg m/s^2

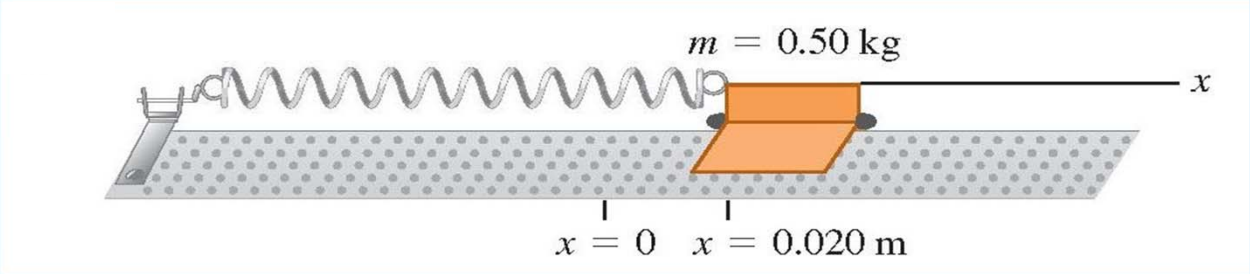
Hooke's law for a spring

$$F = -kX$$

$$k = -\frac{F_x}{x} = -\frac{-6.0 \text{ N}}{0.030 \text{ m}} = 200 \text{ N/m} = 200 \text{ kg/s}^2$$

Harmonic oscillation: Problem

k = 200 kg/s^2



The mass is withdrawn 2 cm and released.
What will be the angular frequency, frequency and period of the oscillations?

$\omega = 2\pi f$

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{200\ \text{kg/s}^2}{0.50\ \text{kg}}} = 20\ \text{rad/s}$$

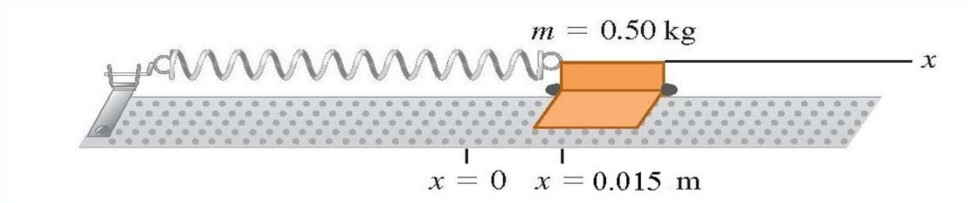
$$f = \frac{\omega}{2\pi} = \frac{20\ \text{rad/s}}{2\pi\ \text{rad/cycle}} = 3.2\ \text{cycle/s} = 3.2\ \text{Hz}$$

$$T = \frac{1}{f} = \frac{1}{3.2\ \text{cycle/s}} = 0.31\ \text{s}$$

Harmonic oscillation: Problem

$k = 200\ kg/s^2$
$\omega = 20\ rad/s$

$t = 0$
$x_0 = 0.015\ m$
$v_0 = +0.40\ m/s$



What is the amplitude and the phase angle ?

$$x = A\cos(\omega t+\phi) \rightarrow x_{max} = A$$

$$v = \frac{dx}{dt} = -\omega A\sin(\omega t+\phi) \rightarrow v_{max} = \omega A$$

$$a = \frac{dv}{dt} = -\omega^2 A\cos(\omega t+\phi) \rightarrow a_{max} = \omega^2 A$$

t = 0

$$x_0 = A\cos\phi$$

$$v_{0x} = -\omega A\sin\phi$$

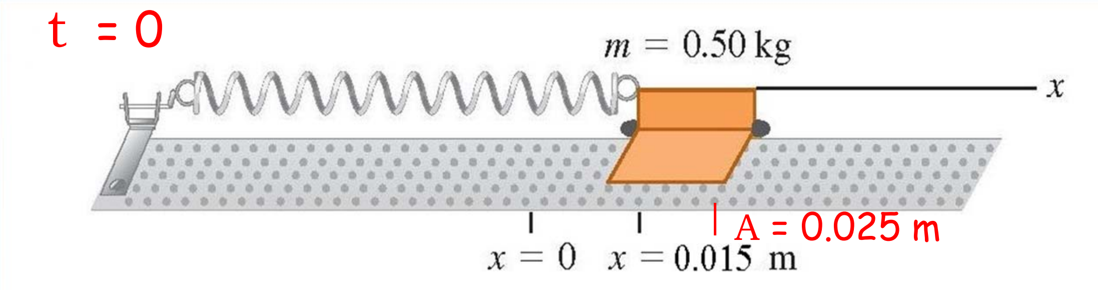
$$\frac{v_{0x}}{x_0} = \frac{-\omega A\sin\phi}{A\cos\phi} = -\omega\tan\phi$$

$$\phi = \arctan\left(-\frac{v_{0x}}{\omega x_0}\right) = \arctan\left(-\frac{0.40\ m/s}{(20\ rad/s)(0.015\ m)}\right) = -53^\circ = -0.93\ rad$$

$$A = x_0 / \cos\phi = 0.015 / \cos(-0.93) = 0.025\ m$$

Harmonic oscillation: Problem

$k = 200\ kg/s^2$
$\omega = 20\ rad/s$
$\phi = -0.93\ rad$
$A = 0.025\ m$



What are the functions for position, velocity and acceleration ?

$$x = A\cos(\omega t+\phi) \longrightarrow x_{max} = A$$

$$v = \frac{dx}{dt} = -\omega A\sin(\omega t+\phi) \longrightarrow v_{max} = \omega A$$

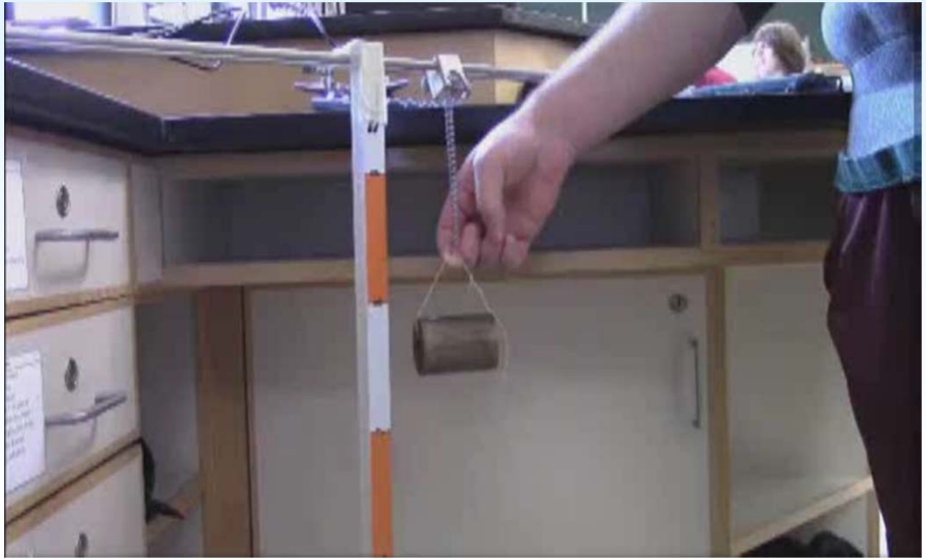
$$a = \frac{dv}{dt} = -\omega^2 A\cos(\omega t+\phi) \longrightarrow a_{max} = \omega^2 A$$

$$x = (0.025\ \text{m})\cos[(20\ \text{rad/s})t - 0.93\ \text{rad}]$$

$$v_x = -(0.50\ \text{m/s})\sin[(20\ \text{rad/s})t - 0.93\ \text{rad}]$$

$$a_x = -(10\ \text{m/s}^2)\cos[(20\ \text{rad/s})t - 0.93\ \text{rad}]$$

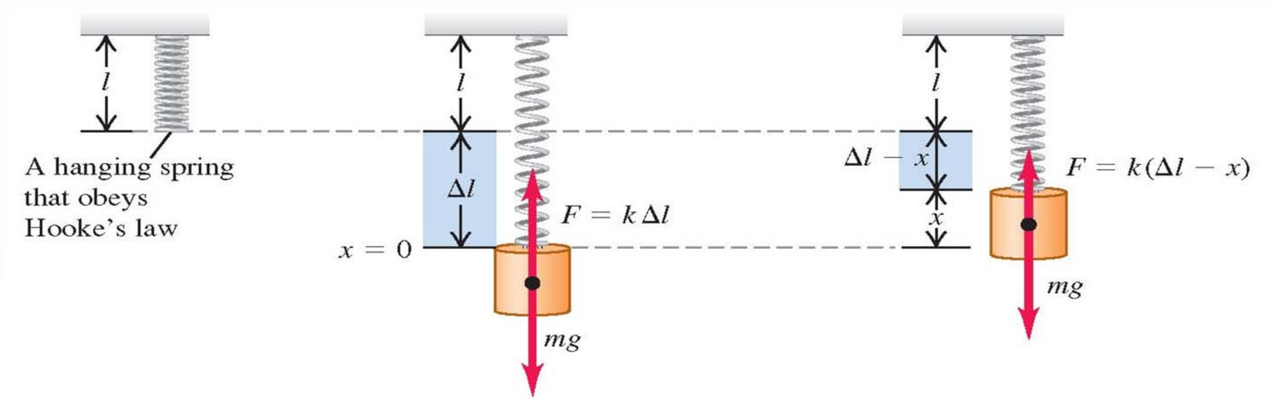
Part 5. Vertical oscillation



Vertical harmonic oscillation

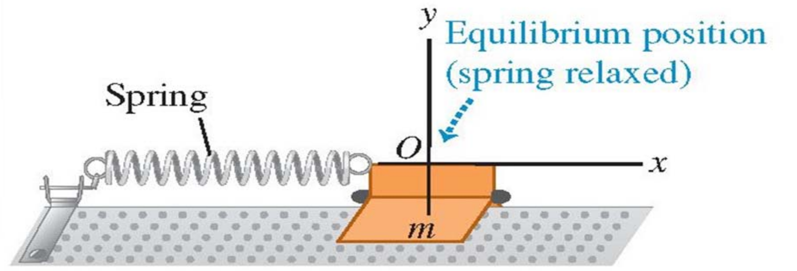
Vertical oscillation

Gravity will stretch the spring to a new eqilibrium position.



Horizontal oscillation

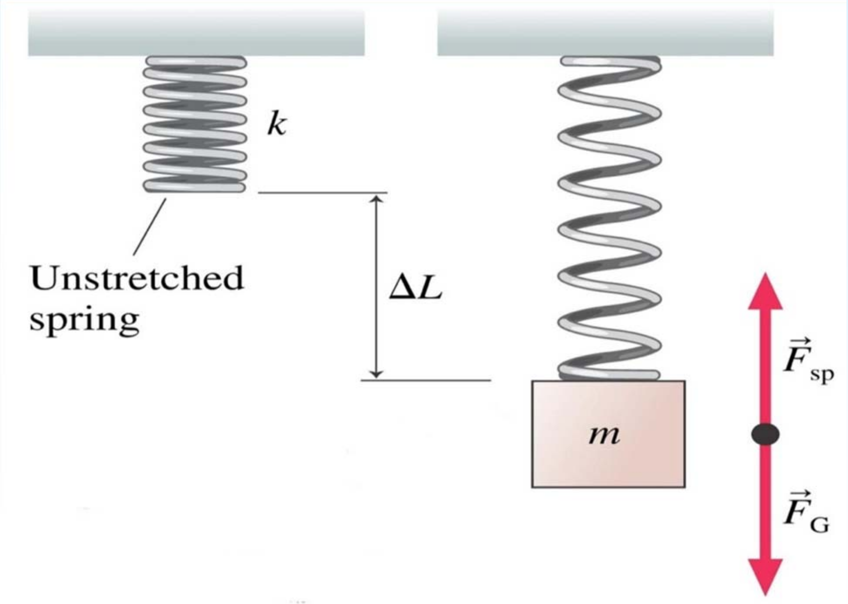
Gravity will not stretch the spring to a new eqilibrium position.



However, the oscillations will be the same.

Vertical harmonic oscillation

Without oscillations: How much is the spring pulled out ?



$$\vec{F}_{total} = \vec{F}_{sp} - \vec{F}_G = k\Delta L - mg$$

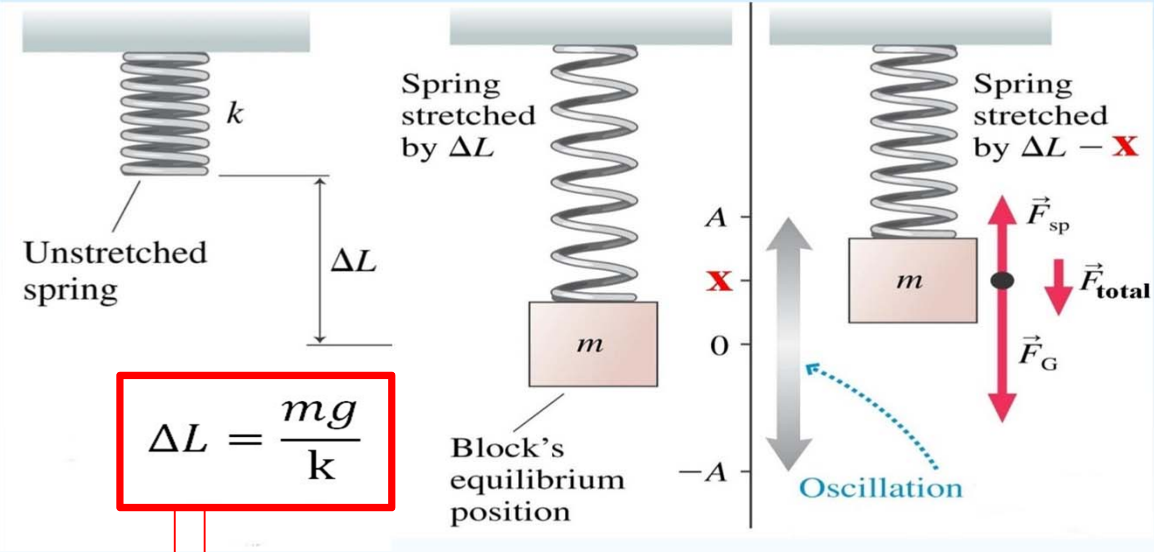
$$\vec{F}_{total} = m\vec{a} = 0$$

$$\Delta L = \frac{mg}{k}$$

Vertical harmonic oscillation

With oscillations:

Add up the forces !



$$\Delta L = \frac{mg}{k}$$

$$\vec{F}_{total} = \vec{F}_{sp} - \vec{F}_G = k(\Delta L - x) - mg \quad = -kx$$

Vertical harmonic oscillation

Hooke's law:

$$\vec{F}_{total} = \vec{F}_{sp} - \vec{F}_G = -kx$$

Newton's law:

$$\vec{F}_{total} = m\vec{a} \neq 0$$

$$-kx = m\vec{a} = m\frac{\partial^2 x}{\partial t^2} \Rightarrow \frac{\partial^2 x}{\partial t^2} + \frac{k}{m}x = 0$$

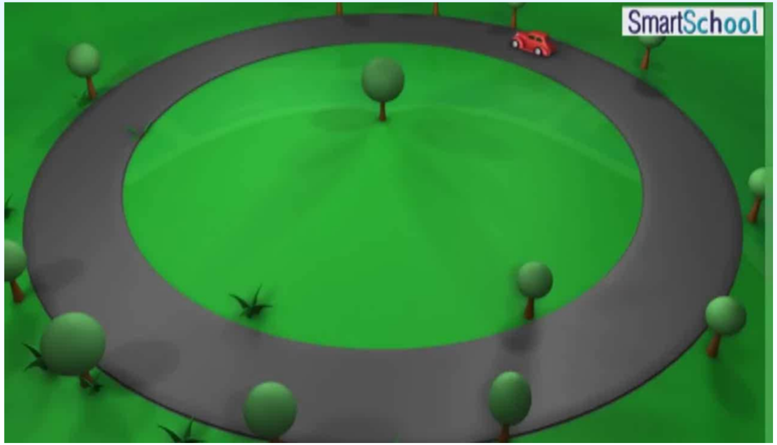
This differential equation has the following solution:

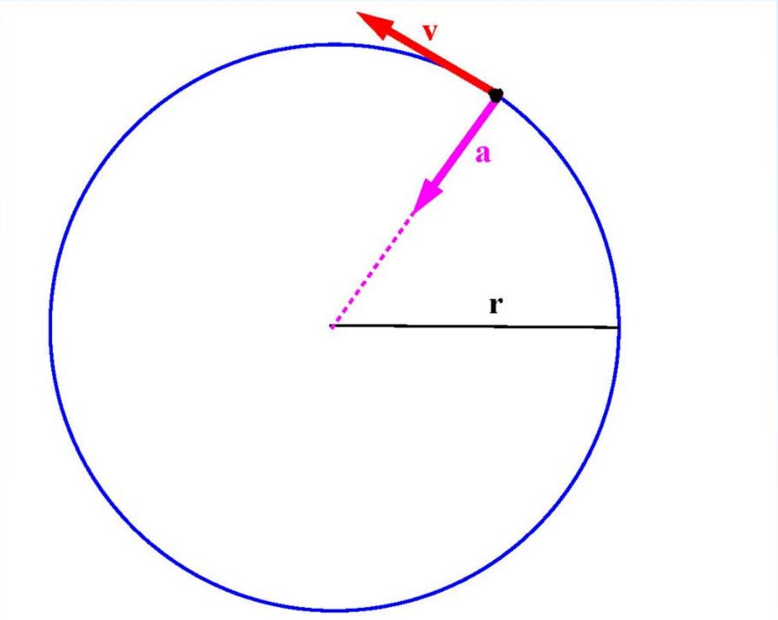
$$x = A\cos(\omega t + \varphi) \qquad \omega = \sqrt{\frac{k}{m}}$$

Part 6. Circular motion and harmonic oscillations

Harmonic oscillation: Circular motion

Description of circular motion if the velocity $|\bar{v}|$ is constant



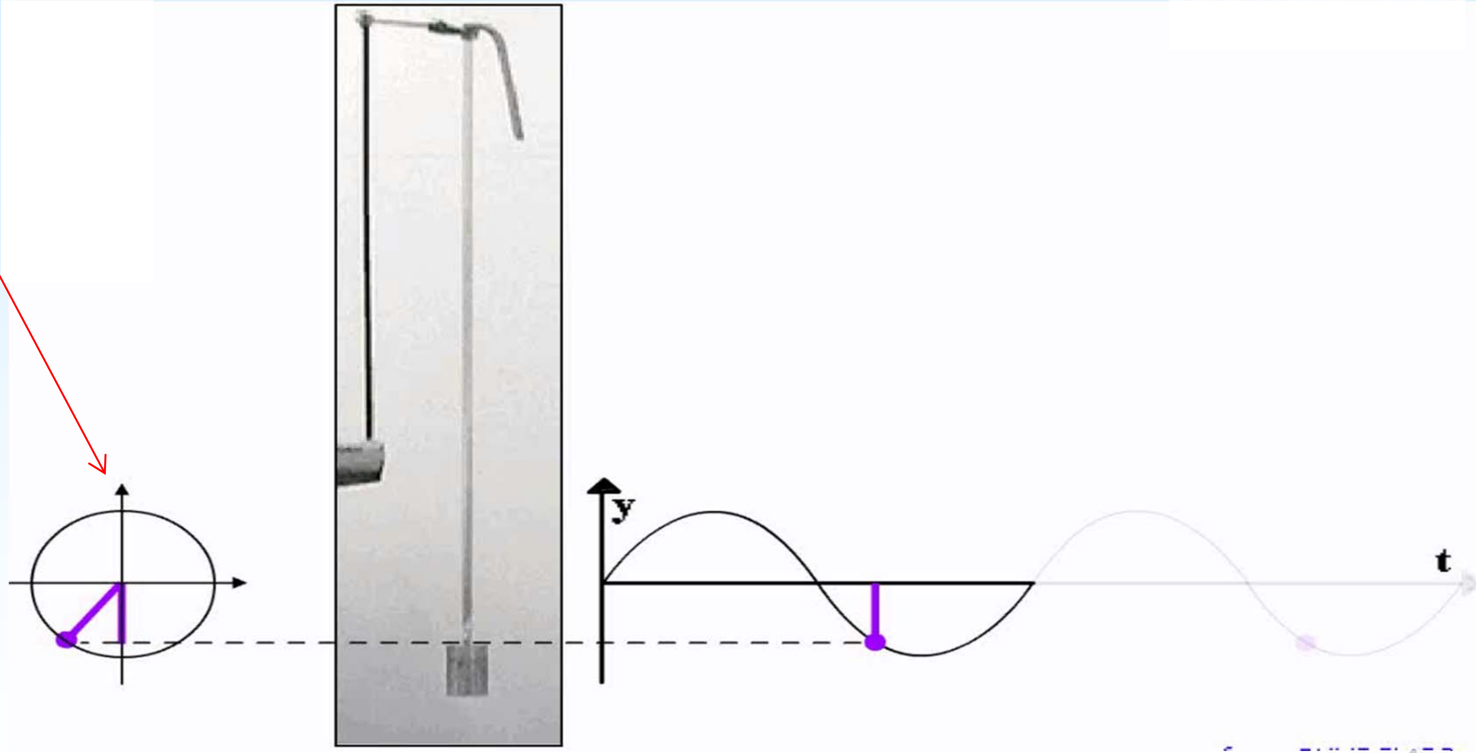


$$v = \text{velocity} = \frac{\text{distance}}{\text{time}} = \frac{\text{circumference}}{\text{time period}} = \frac{2\pi r}{T} = \omega r$$

$$a = \text{acceleration} = v^2/r = \omega^2 r$$

Harmonic oscillation: Circular motion

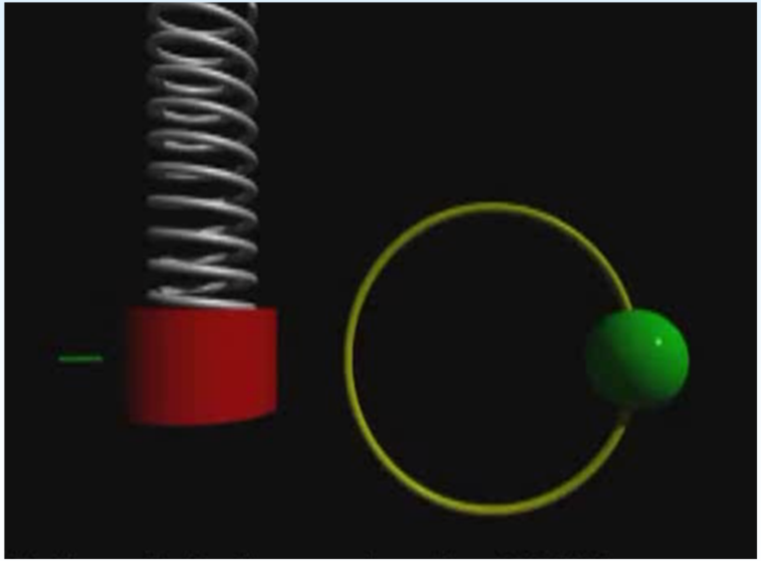
A harmonic oscillation can be described by the y component of a circular motion.



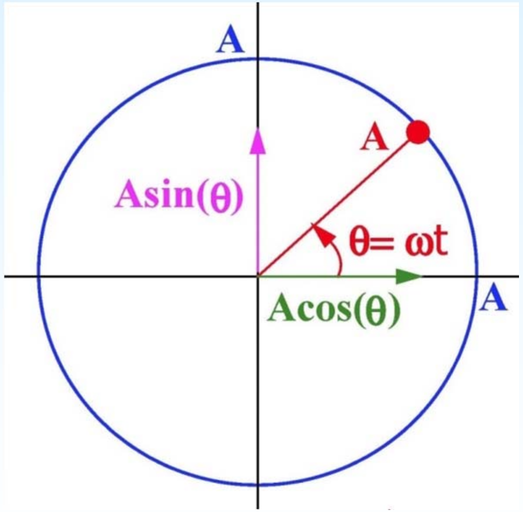
http://www.animations.physics.unsw.edu.au/jw/flash/shm_spring1.swf

Harmonic oscillation: Circular motion

Since harmonic oscillation is described by a sinus function it can also be compared to a circular motion.



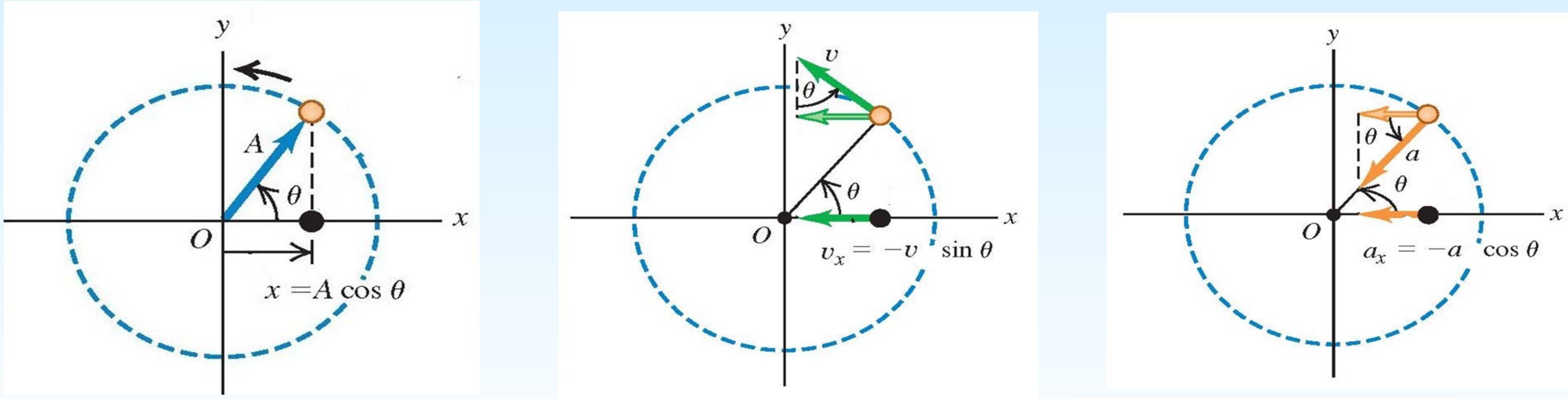
https://www.youtube.com/watch?v=9r0HexjGRE4



$\theta = \omega t$

The angle increases linearly with time

What is x, v and a in the x-direction ?



$x = A\cos\theta$ $\qquad v_x = -v \sin\theta$ $\qquad a_x = -a \cos\theta$

A = radius

Total acceleration: $a = v^2/r = \omega^2 r = \omega^2 A$ $\qquad a_x = -\omega^2 A\cos\theta$

Combine

the acceleration from the discussion about forces

with

the acceleration in circular motion.

Harmonic oscillation: Circular motion

Forces

F = m a
F = -k x

$$a_x = -\frac{k}{m}x$$

Circular Motion

$$x = A\cos\theta$$
$$a_x = -\omega^2 A\cos\theta$$

$$a_x = -\omega^2 x$$

$$\omega = \sqrt{\frac{k}{m}}$$

Simple harmonic motion requires a restoring force that is proportinal to the displacement.

Harmonic oscillation: equations of motion

Movement of a mass hanging from a spring:

$x = 0$ when $t = 0$

Displacement:		$x = A\sin(\omega t) \rightarrow x_{max} = A$
Velocity:	$v = \frac{dx}{dt}$	$v = \omega A\cos(\omega t) \rightarrow v_{max} = \omega A$
Acceleration:	$a = \frac{dv}{dt}$	$a = -\omega^2 A\sin(\omega t) \rightarrow a_{max} = \omega^2 A$

A mass in circular movement:

$x = A$ when $t = 0$

Displacement:		$x = A\cos(\omega t) \rightarrow x_{max} = A$
Velocity:	$v = \frac{dx}{dt}$	$v = -\omega A\sin(\omega t) \rightarrow v_{max} = \omega A$
Acceleration:	$a = \frac{dv}{dt}$	$a = -\omega^2 A\cos(\omega t) \rightarrow a_{max} = \omega^2 A$

Harmonic oscillations in general:

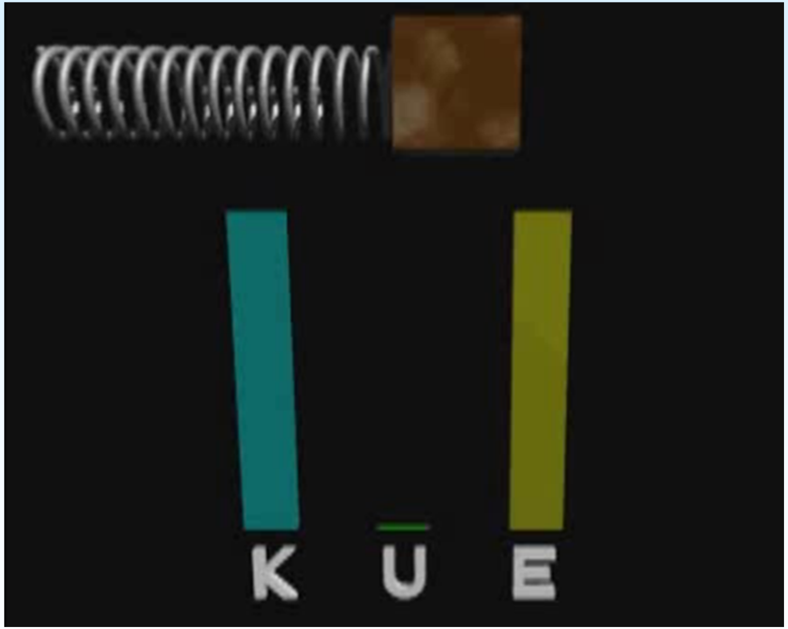
$x = A\cos(\phi)$ when $t = 0$

ϕ = phase angle

(ϕ determines position at $t = 0$)

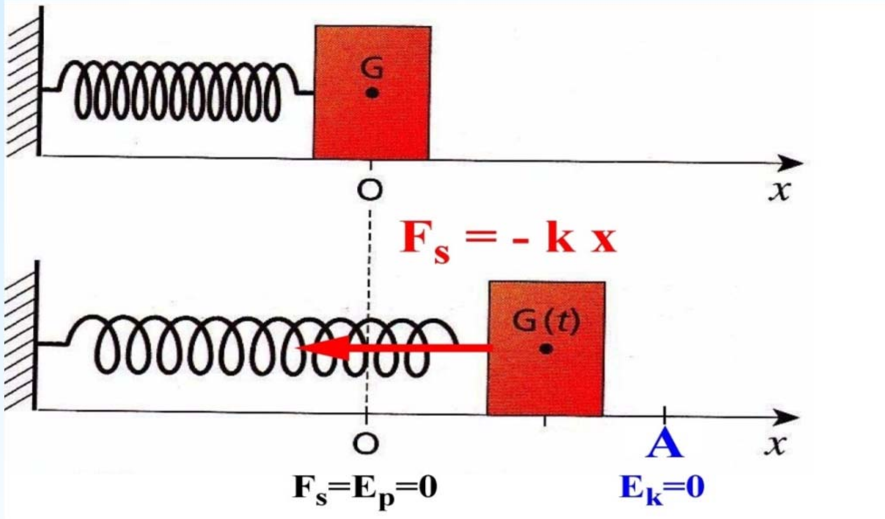
Displacement:		$x = A\cos(\omega t+\phi) \rightarrow x_{max} = A$
Velocity:	$v = \frac{dx}{dt}$	$v = -\omega A\sin(\omega t+\phi) \rightarrow v_{max} = \omega A$
Acceleration:	$a = \frac{dv}{dt}$	$a = -\omega^2 A\cos(\omega t+\phi) \rightarrow a_{max} = \omega^2 A$

Part 7.
Energy and harmonic oscillations



https://www.youtube.com/watch?v=PL5g_IwrC5U

Harmonic oscillation: Energy

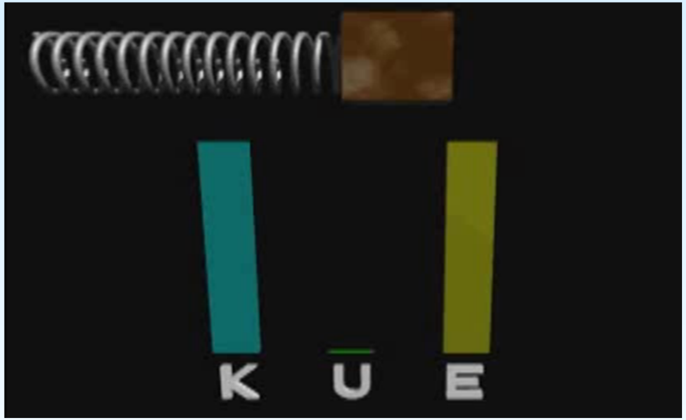


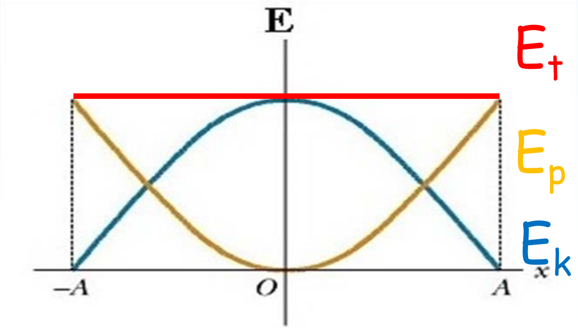
Kinetic energy: $E_k = \frac{mv^2}{2}$ where $v = -\omega A \sin(\omega t)$

Potential energy: $E_p = \frac{kx^2}{2}$ where $x = A\cos(\omega t)$

Total energy: $E_t = E_k + E_p = \frac{kA^2}{2}$ ($E_k = 0$ for $x = A$)

The total mechanical energy is constant





$E_t = \frac{1}{2}kA^2$

E_p

E_k

Harmonic oscillation: Energy

$$x = A\cos(\omega t+\phi)$$

$$v = -\omega A\sin(\omega t+\phi)$$

$$\omega = \sqrt{\frac{k}{m}}$$

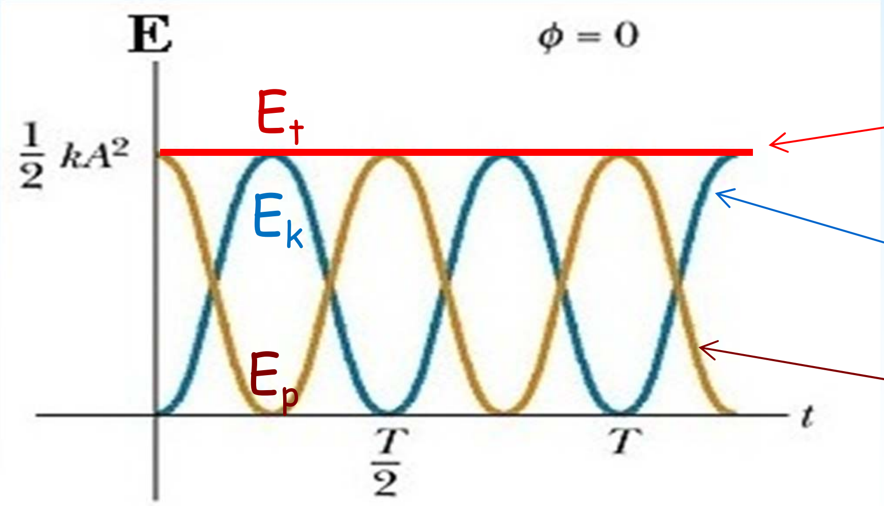
$$E_p = \tfrac{1}{2}kx^2 = \tfrac{1}{2}kA^2\cos^2(\omega t+\phi)$$

$$E_k = \tfrac{1}{2}mv^2 = \tfrac{1}{2}m\omega^2A^2\sin^2(\omega t+\phi) = \tfrac{1}{2}kA^2\sin^2(\omega t+\phi)$$

$$E_t = E_p + E_k = \tfrac{1}{2}kA^2[\cos^2(\omega t+\phi) + \sin^2(\omega t+\phi)] = \tfrac{1}{2}kA^2$$

Harmonic oscillation: Energy

The time dependence of the energy is described by the square of sine and cosine functions:



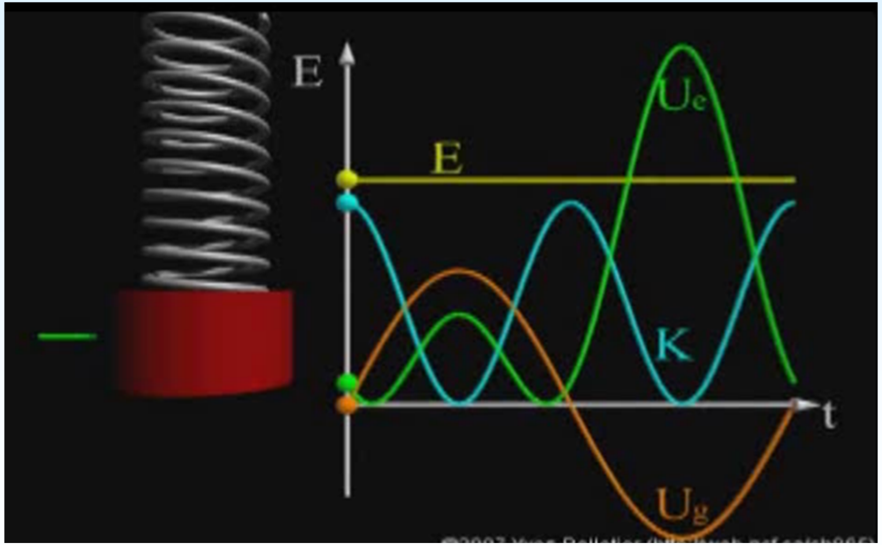
$$E_t = E_p + E_k = \tfrac{1}{2}kA^2$$

$$E_k = \tfrac{1}{2}mv^2 = \tfrac{1}{2}kA^2\sin^2(\omega t)$$

$$E_p = \tfrac{1}{2}kx^2 = \tfrac{1}{2}kA^2\cos^2(\omega t)$$

Harmonic oscillation: Energy

If the oscillation is vertical, a potential energy is also obtained from gravity.



U_e: Elastic potential energy

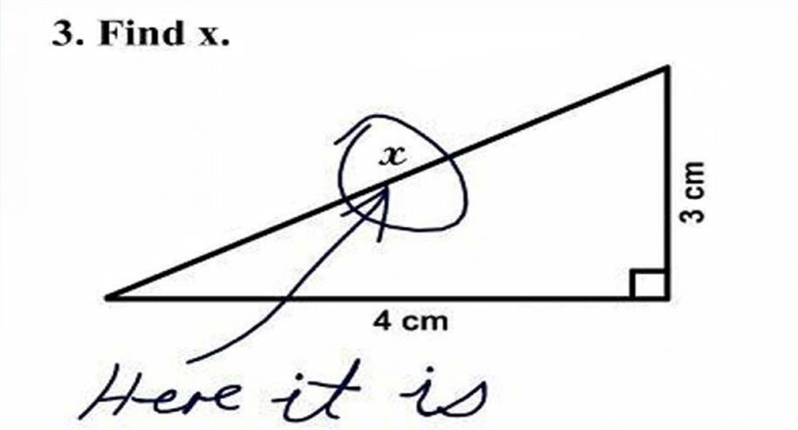
U_g: Potential energy due to gravity

K: Kinetic energy

E: Total mechanical energy

https://www.youtube.com/watch?v=lIPWyY__N2A

Part 8. Problems

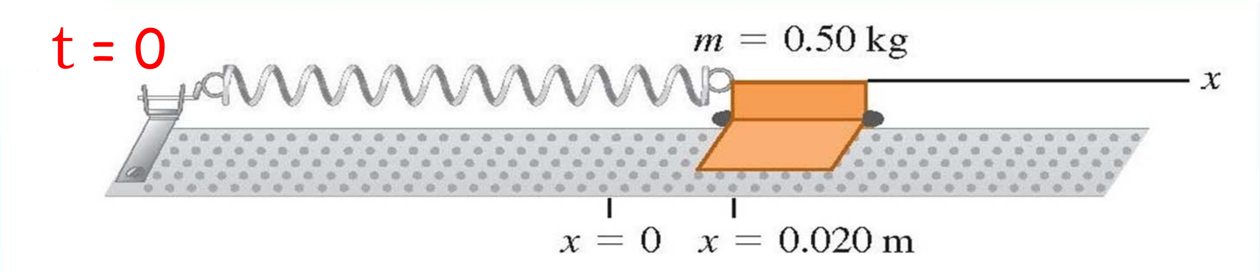


Harmonic oscillation: Problem

$A = 0.020$ m

$k = 200$ N/m

$m = 0.50$ kg



What is v_{max}, a_{max} and ω ?

$$x = A\cos(\omega t + \phi) \longrightarrow x_{max} = A$$

$$v = \frac{dx}{dt} = -\omega A\sin(\omega t + \phi) \longrightarrow v_{max} = \omega A$$

$$a = \frac{dv}{dt} = -\omega^2 A\cos(\omega t + \phi) \longrightarrow a_{max} = \omega^2 A$$

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{200\ \text{kg/s}^2}{0.50\ \text{kg}}} = 20\ \text{rad/s}$$

$$v_{max} = 20 \cdot 0.020 = 0.40\ \text{m/s}$$

$$a_{max} = 20 \cdot 20 \cdot 0.020 = 8\ \text{m/s}^2$$

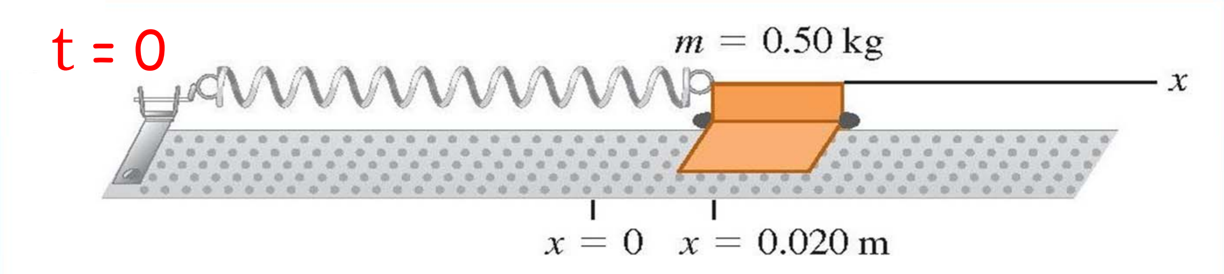
Harmonic oscillation: Problem

$A = 0.020$ m

$k = 200$ N/m

$m = 0.50$ kg

$\omega = 20$ rad/s



What is the phase angle ?

$$x = A\cos(\omega t+\phi) \longrightarrow x_{max} = A$$

$$v = \frac{dx}{dt} = -\omega A\sin(\omega t+\phi) \longrightarrow v_{max} = \omega A$$

$$a = \frac{dv}{dt} = -\omega^2 A\cos(\omega t+\phi) \longrightarrow a_{max} = \omega^2 A$$

Getting the phase angle:

$x = A$ when $t = 0$

$A = A\cos(0 + \phi)$

$\phi = 0$

Harmonic oscillation: Problem

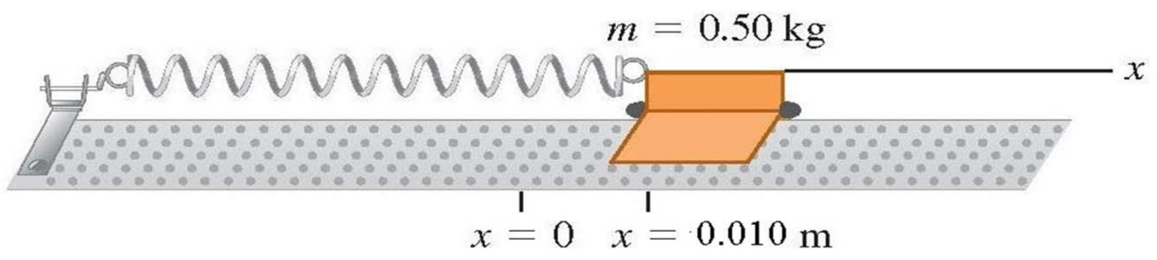
$A = 0.020$ m

$k = 200$ N/m

$m = 0.50$ kg

$\omega = 20$ rad/s

$\phi = 0$



What is v and a when x is halfway in from the maximum position ?

$$x = A\cos(\omega t) \longrightarrow x_{max} = A$$

$$v = \frac{dx}{dt} = -\omega A\sin(\omega t) \longrightarrow v_{max} = \omega A$$

$$a = \frac{dv}{dt} = -\omega^2 A\cos(\omega t) \longrightarrow a_{max} = \omega^2 A$$

$x = A\cos(\omega t)$

$0.010 = 0.020\cos(20t)$

$\omega t = 20t = \mathrm{acos}(0.010/0.020) = 1.047$ rad

$V = -20 \cdot 0.020 \sin(1.047) = -0.35$ m/s

$a = -20^2 \cdot 0.020 \cos(1.047) = -4.0$ m/s^2

Harmonic oscillation: Problem

A = 0.020 m

k = 200 N/m

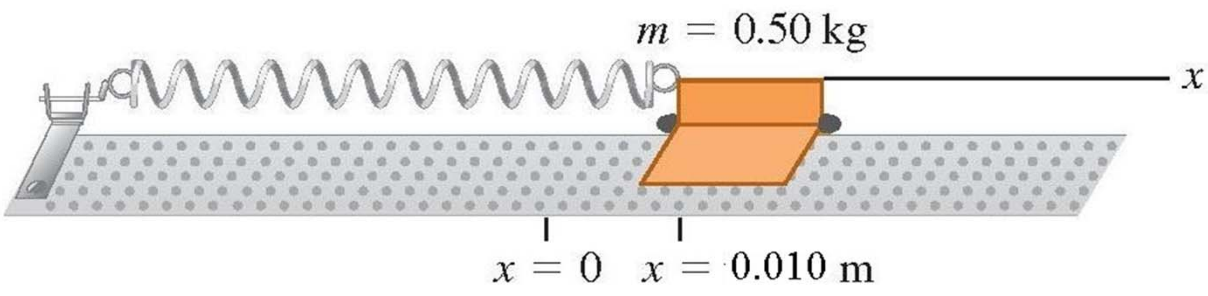
m = 0.50 kg

ω = 20 rad/s

$\phi = 0$

$x = 0.010$ m

$v = -0.35$ m/s



What is the kinetic, potential and total energy ?

Kinetic energy: $E_k = \frac{mv^2}{2}$ where $v = -\omega A \sin(\omega t)$

Potential energy: $E_p = \frac{kx^2}{2}$ where $x = A\cos(\omega t)$

Total energy: $E_t = E_k + E_p = \frac{kA^2}{2}$ ($E_k = 0$ for $x = A$)

$$Ep = \frac{1}{2}kx^2 = \frac{1}{2}(200\ \text{N/m})(0.010\ \text{m})^2 = 0.010\ \text{J}$$

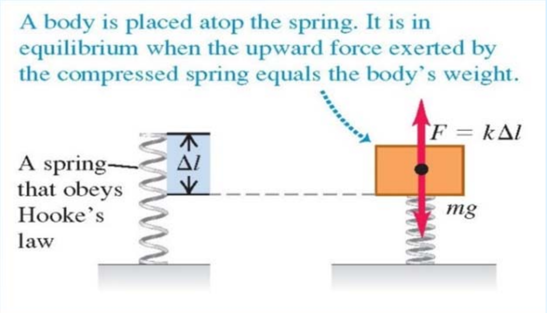
$$Ek = \frac{1}{2}mv_x^2 = \frac{1}{2}(0.50\ \text{kg})(-0.35\ \text{m/s})^2 = 0.030\ \text{J}$$

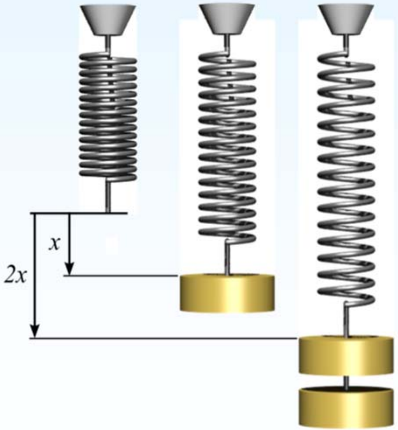
$$E_T = E_p + E_k = 0.040\ J$$

Harmonic oscillation: Problem

Assume the following: A car has a mass of 1000 kg. A drivers weight is F = 980 N and causes the shock absorbers to drop by 2.8 cm. The car drives over a bump and begins to swing by harmonic oscillation.

What will be the period and frequency ?





$F_x = -kx$
$f = 1/T$
$\omega = 2\pi f$

$$\omega = \sqrt{\frac{k}{m}}$$

$$k = -\frac{F_x}{x} = -\frac{980\ \text{N}}{-0.028\ \text{m}} = 3.5 \times 10^4\ \text{kg/s}^2$$

The person's mass is $w/g = (980\ \text{N})/(9.8\ \text{m/s}^2) = 100\ \text{kg}$. The *total* oscillating mass is $m = 1000\ \text{kg} + 100\ \text{kg} = 1100\ \text{kg}$. The period T is

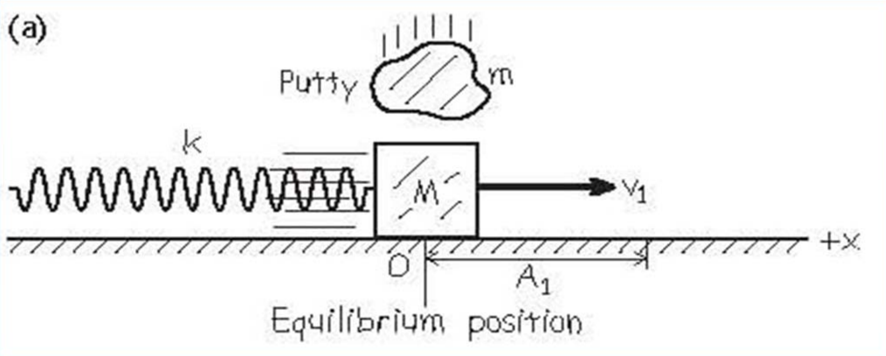
$$T = \frac{1}{f} = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{1100\ \text{kg}}{3.5 \times 10^4\ \text{kg/s}^2}} = 1.11\ \text{s}$$

The frequency is $f = 1/T = 1/(1.11\ \text{s}) = 0.90\ \text{Hz}$.

Harmonic oscillation: Problem

A lump of clay with the mass m falls on a moving mass M at the equilibrium position.

Calculate the new period T_2 !
Give the result as a function of
k, m, M !



$f = 1/T$
$\omega = 2\pi f$

$$T = \frac{1}{f} = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$$

$$\omega = \sqrt{\frac{k}{m}}$$

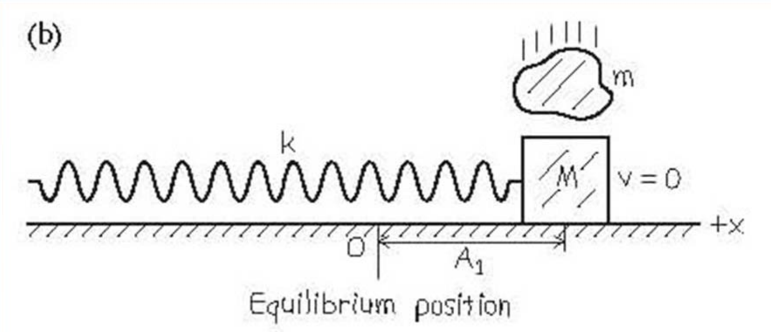
The new period T_2:

$$T_2 = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{M+m}{k}}$$

Harmonic oscillation: Problem

A lump of clay with the mass m falls on a moving mass M at the maximum position.

Calculate the new period T_2 and the new amplitude A_2!



$f = 1/T$

$\omega = 2\pi f$

$$\omega = \sqrt{\frac{k}{m}}$$

$$T = \frac{1}{f} = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$$

$$T_2 = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{M+m}{k}}$$

For x = A the kinetic energy = 0:

$$E_{t1} = 0 + E_{p1} = \frac{1}{2}kA_1^2$$

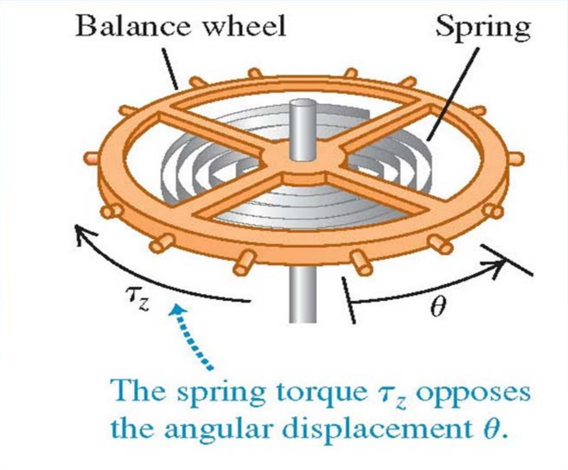
$$E_{t2} = 0 + E_{p2} = \frac{1}{2}kA_2^2$$

The total energy is conserved: $E_{t1} = E_{t2}$ och $A_2 = A_1$

Part 9. Harmonic angular motion

The Henry Graves supercomplication
Value: 206 million kronor

The spring in a watch is a harmonic oscillator.



$$\theta = \Theta \cos(\omega t + \phi)$$

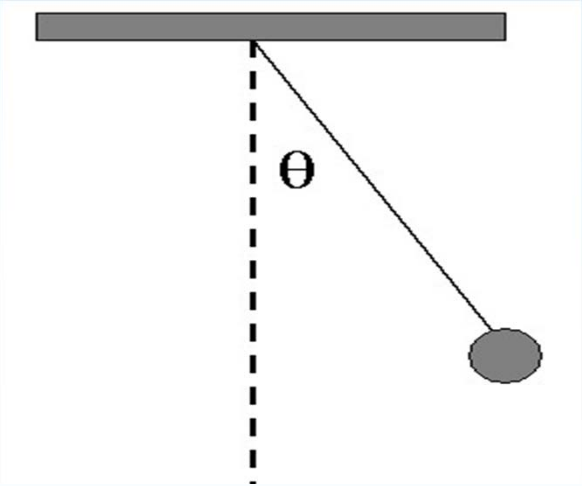
Part 10. The pendulum

Foucault's pendulum

Demonstrates the earth's rotation

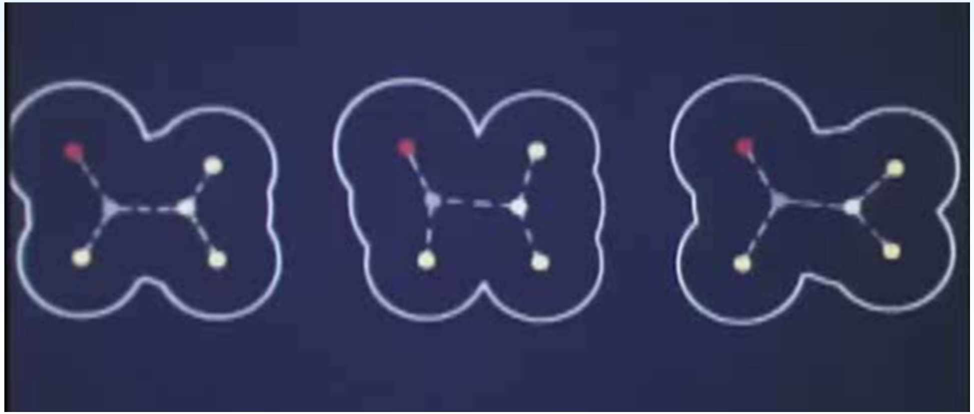
Harmonic oscillation: Pendulum

The pendulum is a harmonic oscillator.



$$\theta = \Theta \cos(\omega t + \phi)$$

Part 11. The vibration of molecules

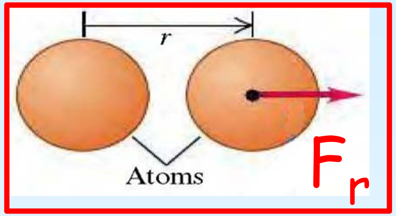


https://www.youtube.com/watch?v=3RqEIr8NtMI

Harmonic oscillation: Molecules

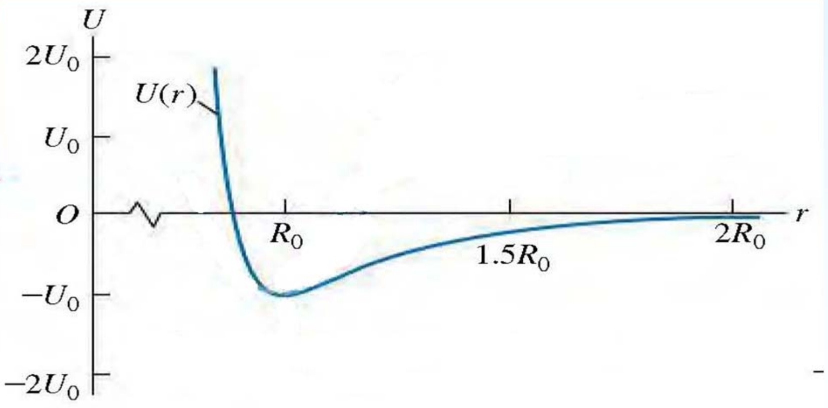
Potential energy (U)

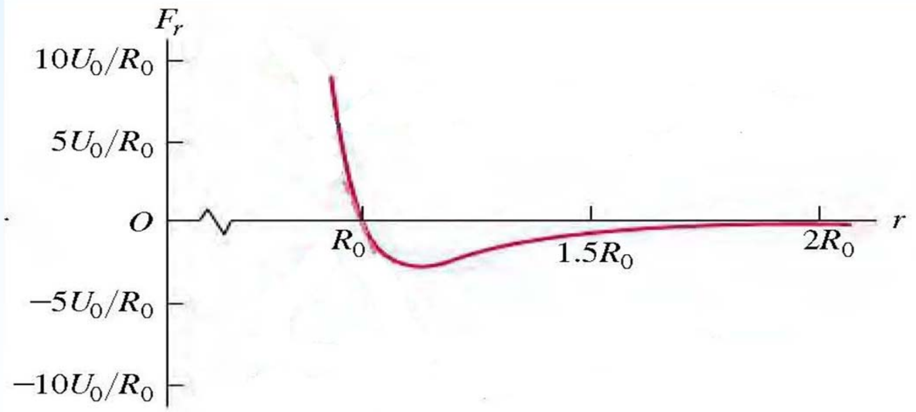
$$U = U_0\left[\left(\frac{R_0}{r}\right)^{12} - 2\left(\frac{R_0}{r}\right)^{6}\right]$$



Force between two atoms (F_r)

$$F_r = -\frac{dU}{dr} = U_0\left[\frac{12R_0^{12}}{r^{13}} - 2\frac{6R_0^{6}}{r^{7}}\right] = 12\frac{U_0}{R_0}\left[\left(\frac{R_0}{r}\right)^{13} - \left(\frac{R_0}{r}\right)^{7}\right]$$



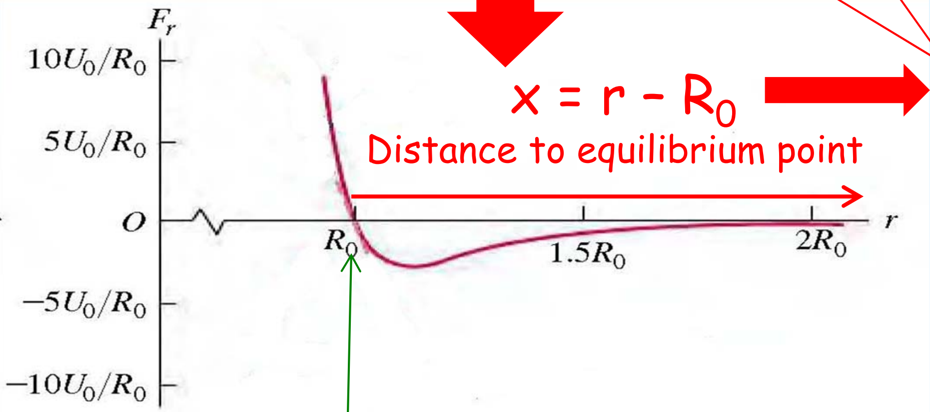


Harmonic oscillation: Molecules

Force between two atoms (F_r)

$$F_r = -\frac{dU}{dr} = U_0\left[\frac{12R_0^{12}}{r^{13}} - 2\frac{6R_0^{6}}{r^{7}}\right] = 12\frac{U_0}{R_0}\left[\left(\frac{R_0}{r}\right)^{13} - \left(\frac{R_0}{r}\right)^{7}\right]$$

$$F_r = 12\frac{U_0}{R_0}\left[\left(\frac{R_0}{R_0+x}\right)^{13} - \left(\frac{R_0}{R_0+x}\right)^{7}\right]$$
$$= 12\frac{U_0}{R_0}\left[\frac{1}{(1+x/R_0)^{13}} - \frac{1}{(1+x/R_0)^{7}}\right]$$



$x = r - R_0$ → $r = R_0 + x$

Distance to equilibrium point

Now simplify this !

The equilibrium point is at $r = R_0$ since then U is at minimum and F = 0

Mathematics: The Binomial Theorem

$$(1+u)^n = 1 + nu + \frac{n(n-1)}{2!}u^2 + \frac{n(n-1)(n-2)}{3!}u^3 + \cdots$$

If u is small one can use the beginning of the series as an approximation:

$$(1+0.001)^{13} = 1.013078......$$

$$(1+0.001)^{13} \approx 1 + 13 \cdot 0.001 = 1.013$$

Harmonic oscillation: Molecules

Now simplify this:

$$F_r = 12\frac{U_0}{R_0}\left[\left(\frac{R_0}{R_0+x}\right)^{13} - \left(\frac{R_0}{R_0+x}\right)^{7}\right] = 12\frac{U_0}{R_0}\left[\frac{1}{(1+x/R_0)^{13}} - \frac{1}{(1+x/R_0)^{7}}\right]$$

with $(1 + u)^n = 1 + nu$

Assume that the vibrations are small so that x/R_0 is small !

We can then use the Binomial Theorem:

$$\frac{1}{(1 + x/R_0)^{13}} = (1 + x/R_0)^{-13} \approx 1 + (-13)\frac{x}{R_0}$$

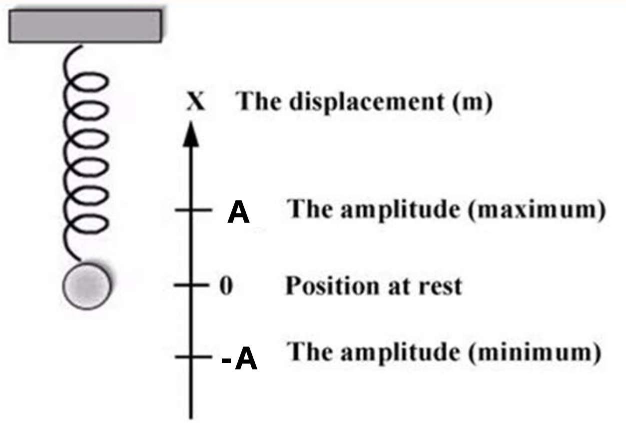
$$\frac{1}{(1 + x/R_0)^{7}} = (1 + x/R_0)^{-7} \approx 1 + (-7)\frac{x}{R_0}$$

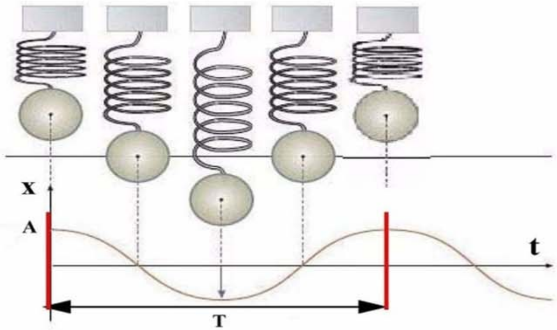
$$F_r \approx 12\frac{U_0}{R_0}\left[\left(1 + (-13)\frac{x}{R_0}\right) - \left(1 + (-7)\frac{x}{R_0}\right)\right] = -\left(\frac{72U_0}{R_0^2}\right)x$$

This is just Hooke's law, with force constant $k = 72U_0/R_0^2$

Part 12. Summary

Harmonic oscillation: Summary





$\phi = \mathrm{acos}(x_0 / A) = \mathrm{acos}(A/A) = 0$

x	The displacement (m)
A	The amplitude (m)
t	Time (s)
T	Period (s)
f	Frequency (Hz) = 1 / T
ω	Angular Frequency (Hz) = 2π / T = 2 π f

$x = A\cos(\omega t+\phi) \longrightarrow x_{max} = A$

$v = \frac{dx}{dt} = -\omega A\sin(\omega t+\phi) \longrightarrow v_{max} = \omega A$

$a = \frac{dv}{dt} = -\omega^2 A\cos(\omega t+\phi) \longrightarrow a_{max} = \omega^2 A$

Harmonic oscillation: Summary

Harmonic oscillations in a spring are described by the equation

$$\frac{\partial^2 x}{\partial t^2} + \frac{k}{m}x = 0$$

if F = -kX

which has the solution

$$x = A\cos(\omega t + \varphi)$$

$$\omega = \sqrt{\frac{k}{m}}$$

Kinetic energy: $E_k = \frac{1}{2}mv^2 = \frac{1}{2}kA^2\sin^2(\omega t)$

Potential energy: $E_p = \frac{1}{2}kx^2 = \frac{1}{2}kA^2\cos^2(\omega t)$

Total energy: $E_t = E_p + E_k = \frac{1}{2}kA^2$