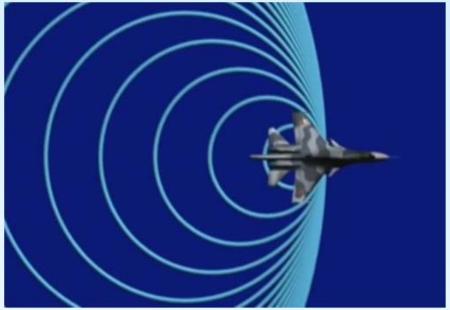
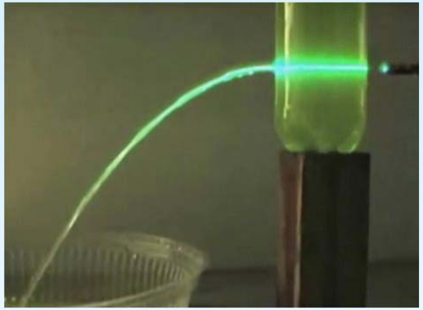
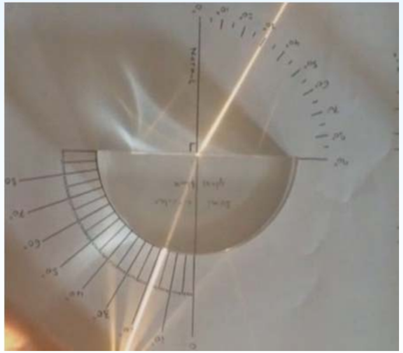
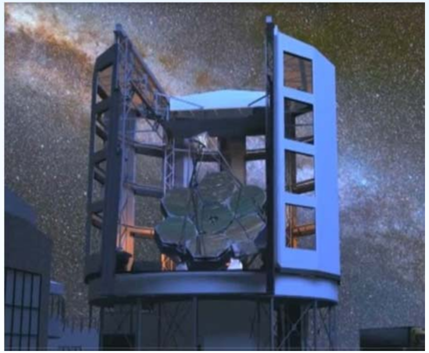
Wavemechanics and optics





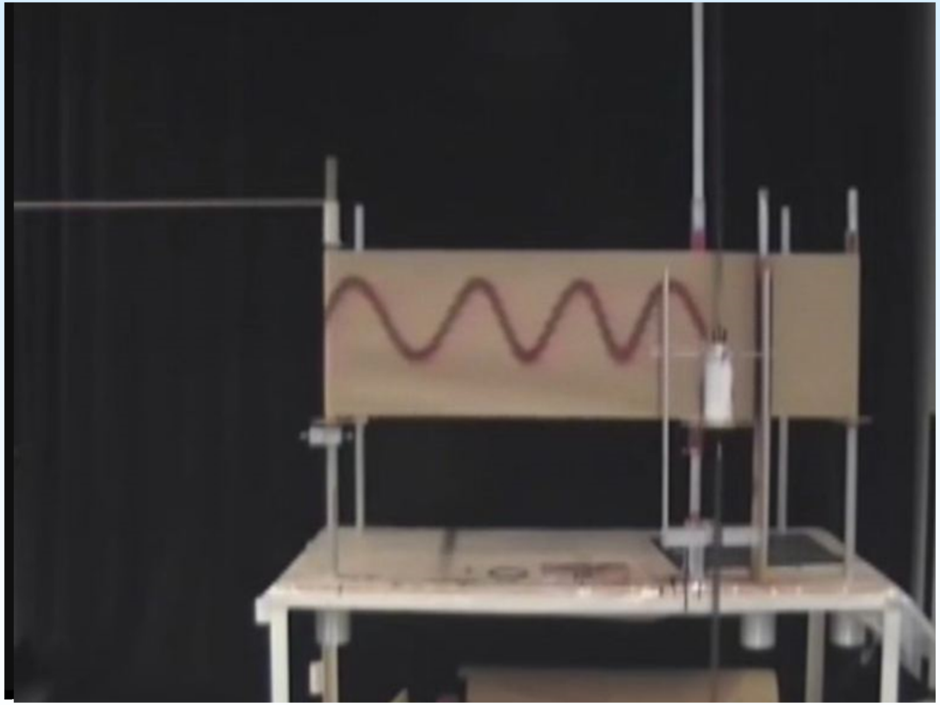




Chapter 14 – Harmonic oscillation

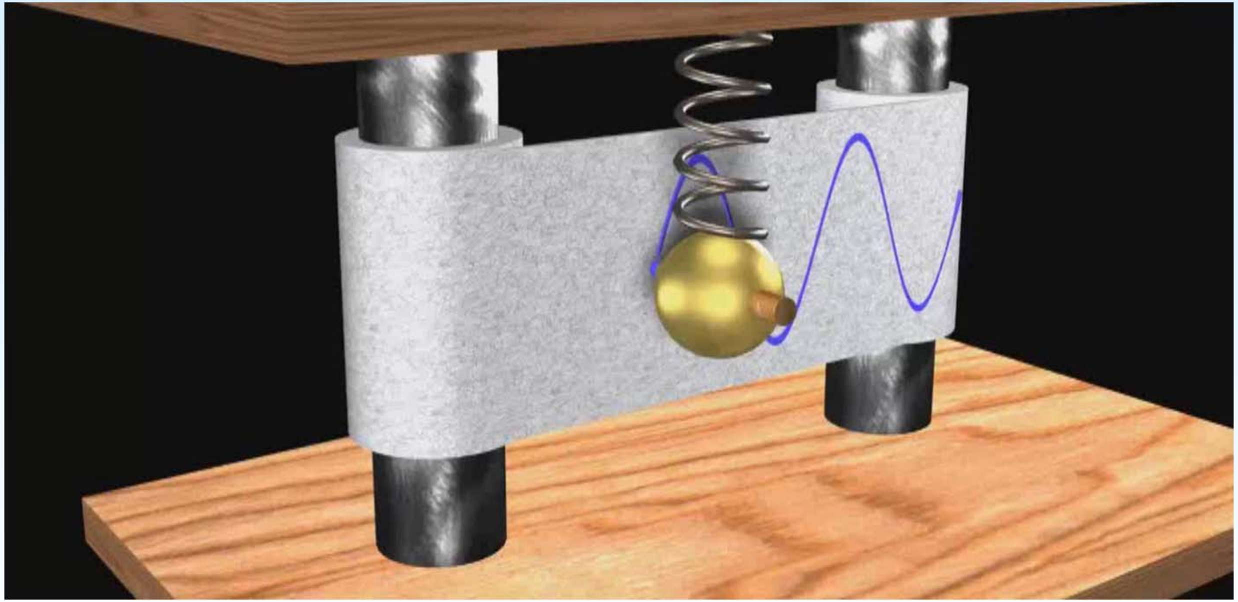
Harmonic oscillation: Experiment

An experiment to find a mathematical description of harmonic oscillation:



https://www.youtube.com/watch?v=p9uhmjbZn-c

Harmonic oscillation: Experiment

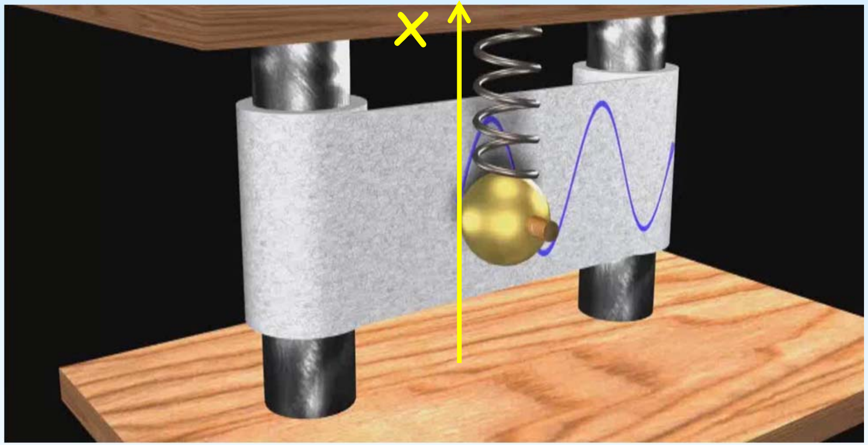


Conclusion: Harmonic oscillation can be described by the function:

$$x = A \sin(Bt + C)$$

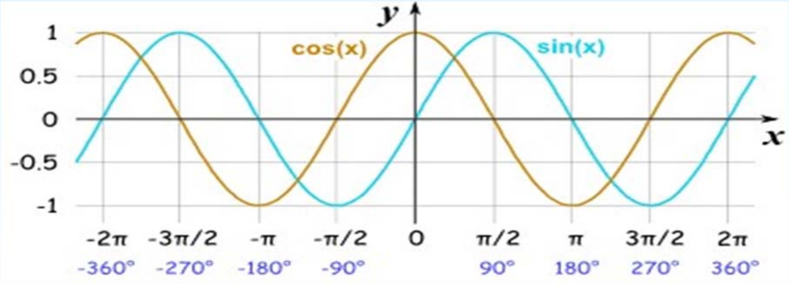
if t is the time and A, B and C are constants that describes the motion.

Harmonic oscillation: Notation



$$x = A\sin(Bt + C) \quad \text{or}$$

$$x = A\cos(Bt + C - \pi/2)$$



x : Vertical displacement. Unit: meter

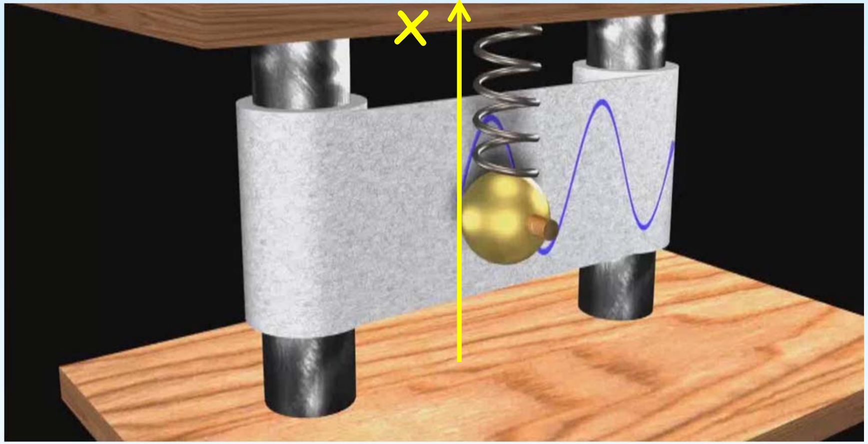
t : Time. Unit: second

A : Amplitude (maximum displacement). Unit: meter

B = ω : Angular frequency (the number of oscillations per second times 2π).
Unit: Radians per second

C = ϕ : Phase angle (determines the position at time = 0). Unit: radians

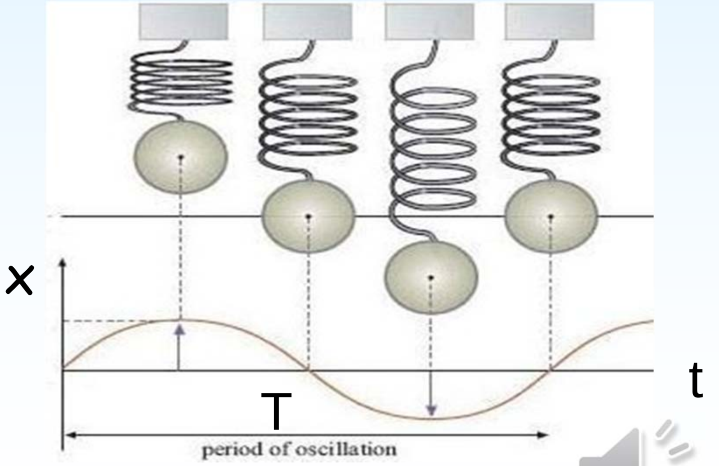
Harmonic oscillation: Notation



$x = A \sin(\omega t + \phi')$

or

$x = A \cos(\omega t + \phi)$



T: Period = the time it takes for the weight to go up and down. Unit: second

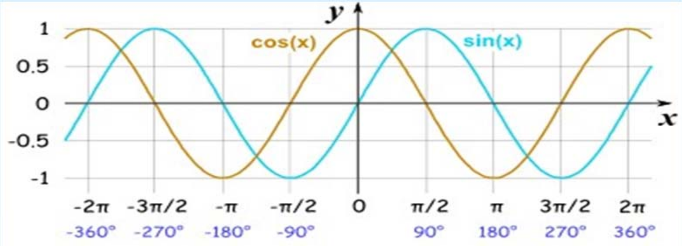
f: Frequency = the number of periods per second. Unit: 1/second = Hz

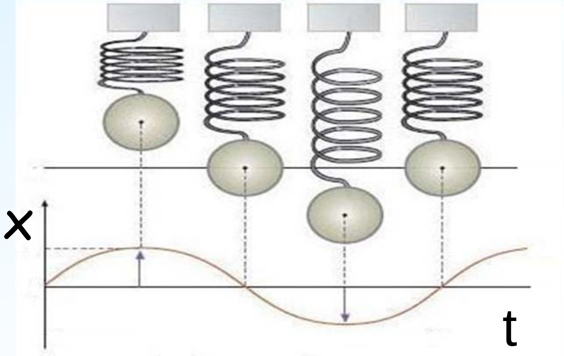
$f = 1 / T$ $\omega = 2\pi f$

Harmonic oscillation: Phase angle

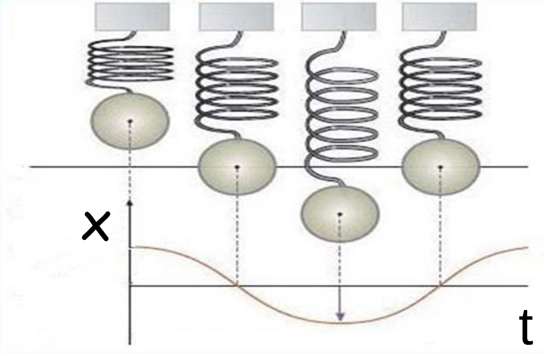
$x = A \sin(\omega t + \phi')$ or $x = A \cos(\omega t + \phi)$

The phase angle (ϕ) determines the position at time = 0 since then $x = A\sin(\phi')$ or $x = A\cos(\phi)$

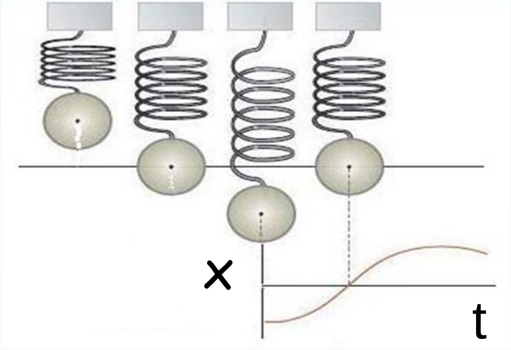


x

t
0

$X = A \sin(\omega t)$
$X = A \cos(\omega t - \pi/2)$

x

t
0

$X = A \cos(\omega t)$
$X = A \sin(\omega t + \pi/2)$

x

t
0

$X = A \cos(\omega t + \pi)$
$X = A \sin(\omega t - \pi/2)$

Harmonic oscillation: velocity & acceleration

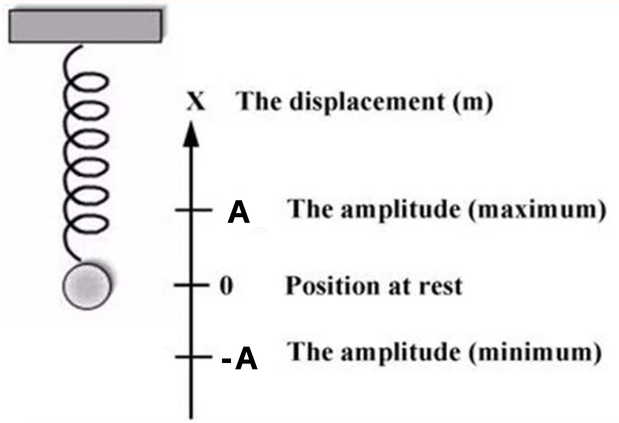
We now have a mathematical description of the displacement.

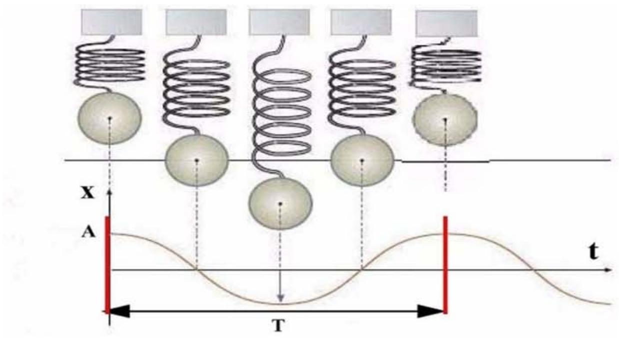
What is the velocity and acceleration ?

$$v(t) = \frac{dx}{dt}$$

$$a(t) = \frac{dv}{dt}$$

Harmonic oscillation: Summary





$\phi = \mathrm{acos}(x_0 / A) = \mathrm{acos}(A/A) = 0$

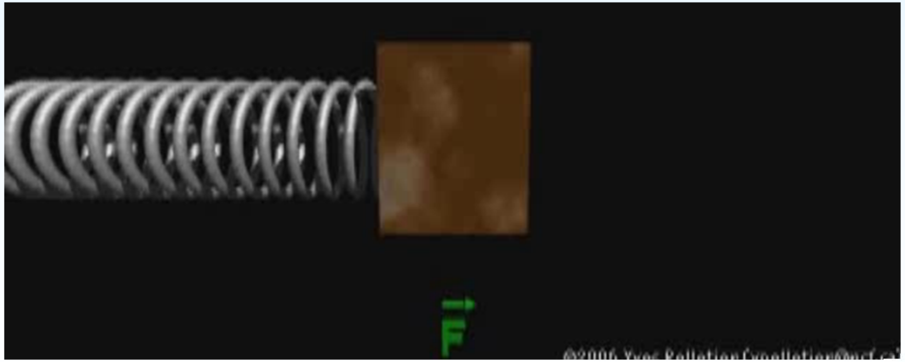
- x The displacement (m)
- A The amplitude (m)
- t Time (s)
- T Period (s)
- f Frequency (Hz) = 1 / T
- ω Angular Frequency (Hz) = 2π / T = 2 π f

$$x = A\cos(\omega t+\phi) \longrightarrow x_{max} = A$$

$$v = \frac{dx}{dt} = -\omega A\sin(\omega t+\phi) \longrightarrow v_{max} = \omega A$$

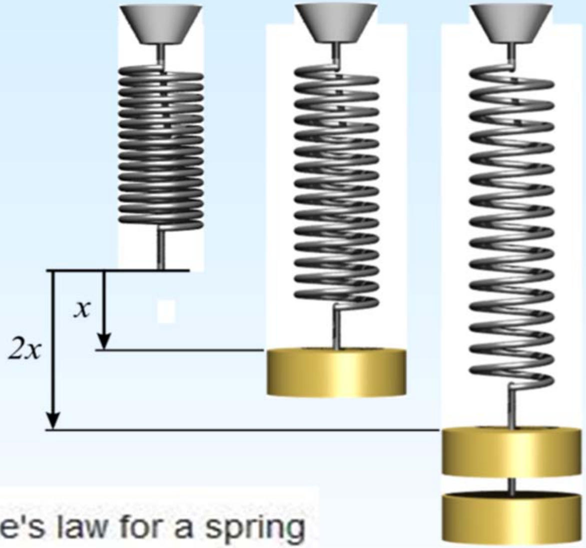
$$a = \frac{dv}{dt} = -\omega^2 A\cos(\omega t+\phi) \longrightarrow a_{max} = \omega^2 A$$

Springs, Hooke's law & Forces



https://www.youtube.com/watch?v=_ca770YbeZw

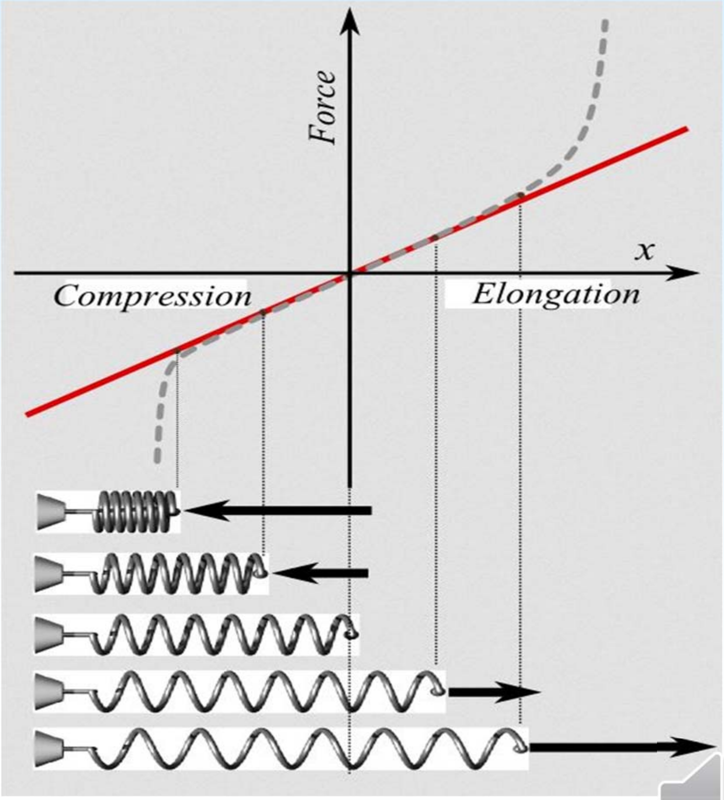
Harmonic oscillation: The spring



Hooke's law for a spring

$$F = -kX$$

k = spring constant
which describes how stiff the spring is.



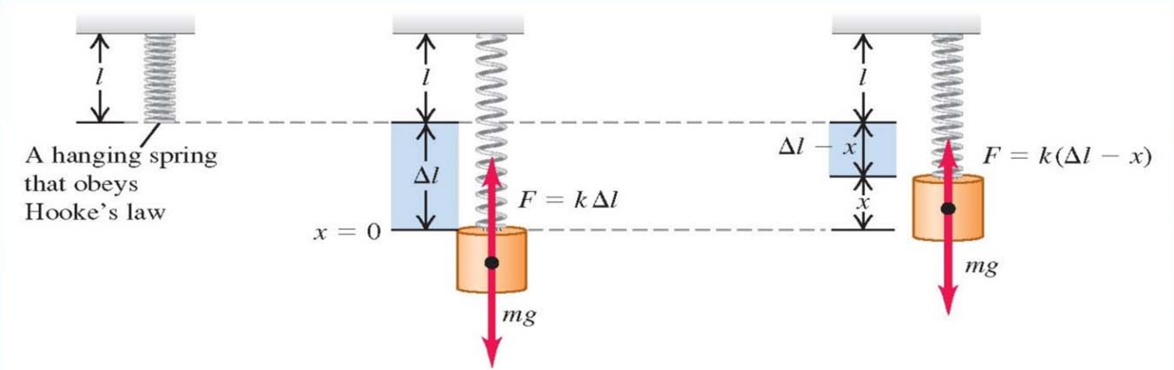
Harmonic oscillation: Forces

Newton's first law of motion: A body acted on by no net force moves with constant velocity (which may be zero) and zero acceleration.

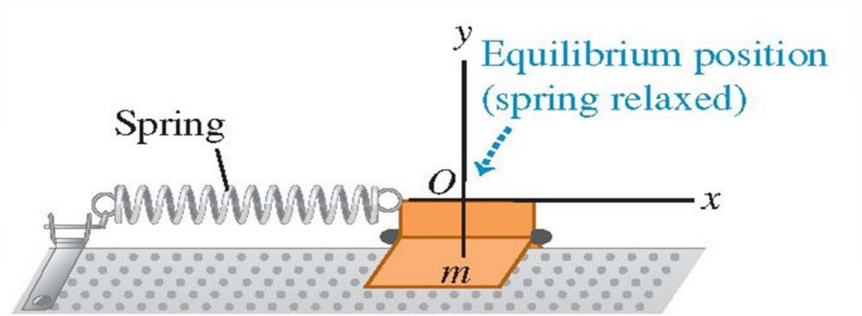
Newton's second law of motion: If a net external force acts on a body, the body accelerates. The direction of acceleration is the same as the direction of the net force. The mass of the body times the acceleration of the body equals the net force vector.

$$\sum \vec{F} = m\vec{a} \qquad \text{(Newton's second law of motion)}$$

Harmonic oscillation with a spring



Vertical oscillation
Gravity will stretch the spring to a new eqilibrium position.



Horizontal oscillation
Gravity will not stretch the spring to a new eqilibrium position.

However, the oscillations will be the same.

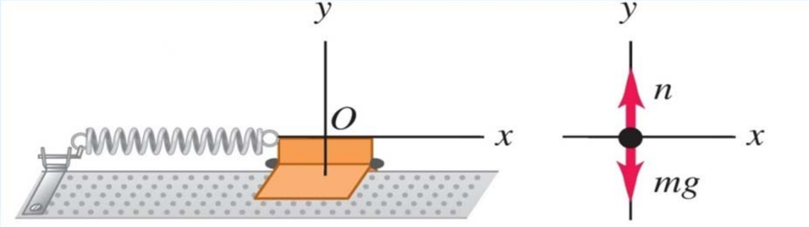
Harmonic oscillation with a spring

Horizontal oscillation on an airbed.

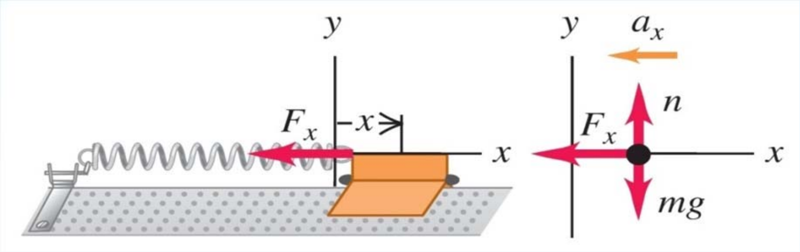
https://www.youtube.com/watch?v=9nLedU7qvvw

Harmonic oscillation: Forces

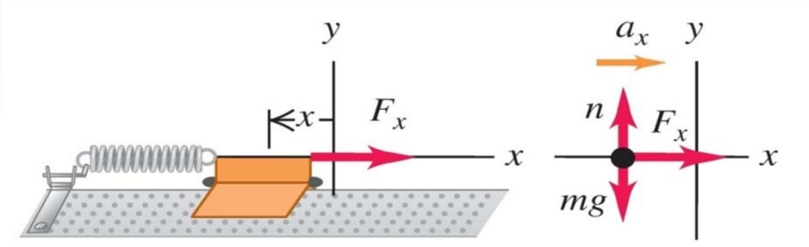
$x = 0 \quad F_{total} = 0 \quad a_x = 0$



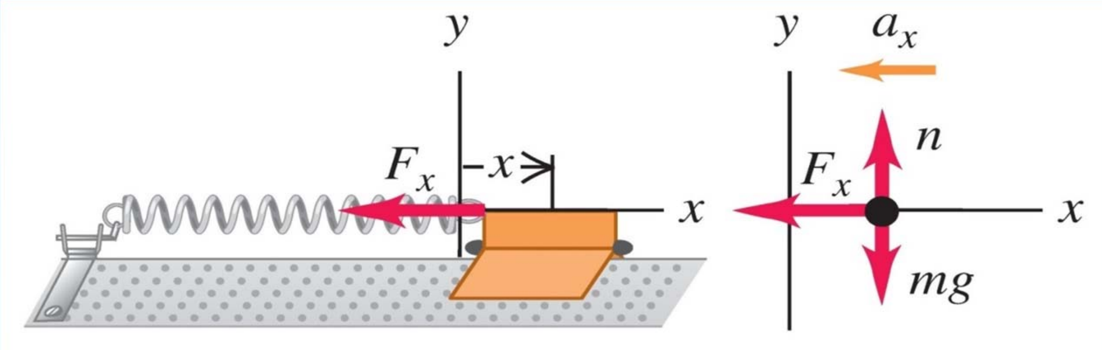
$x > 0 \quad F_{total} < 0 \quad a_x < 0$



$x < 0 \quad F_{total} > 0 \quad a_x > 0$



Harmonic oscillation: Forces



$$F_x = -kx \quad \text{(restoring force exerted by an ideal spring)}$$

$$\sum \vec{F} = m\vec{a} \quad \text{(Newton's second law of motion)}$$

$$a_x = \frac{d^2x}{dt^2} = -\frac{k}{m}x \quad \text{(simple harmonic motion)}$$

Harmonic oscillation: Forces

Old formulas:

$x = A\cos(\omega t+\phi)$

$v = -\omega A\sin(\omega t+\phi)$

$a = -\omega^2 A\cos(\omega t+\phi)$

$\Rightarrow$ $a_x = -\omega^2 x$

New formula:

$$a_x = \frac{d^2x}{dt^2} = -\frac{k}{m}x \qquad \text{(simple harmonic motion)}$$

Combine old and new:

$$-\omega^2 x = -\frac{k}{m}x$$

$$\omega = \sqrt{\frac{k}{m}}$$

The frequency depends on two variables:

1. The spring constant
2. The mass

Harmonic oscillation: Forces

You can look at the oscillations in a different way

$$a_x = \frac{d^2x}{dt^2} = -\frac{k}{m}x \qquad \text{(simple harmonic motion)}$$

$$\frac{\partial^2 x}{\partial t^2} + \frac{k}{m}x = 0$$

This is a differential equation that has the solution

$$x = A\cos(\omega t + \varphi)$$

$$\omega = \sqrt{\frac{k}{m}}$$

$$-\omega^2 A\cos(\omega t + \varphi) + \frac{k}{m} A\cos(\omega t + \varphi) = 0$$

$$-\omega^2 A\cos(\omega t + \varphi) + \omega^2 A\cos(\omega t + \varphi) = 0$$

Harmonic oscillation with a spring

$$\omega = \sqrt{\frac{k}{m}}$$

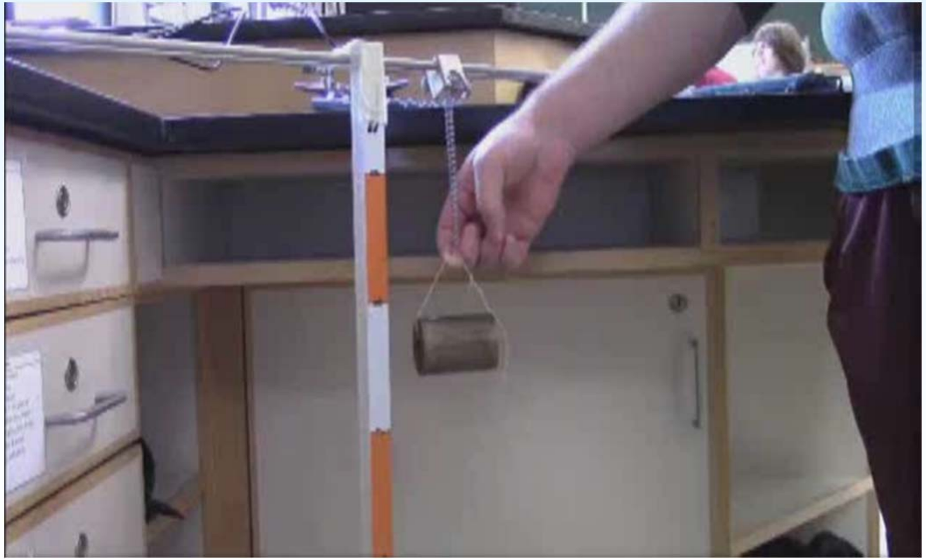
Increase the mass

The frequency decreases

Increase the spring constant

The frequency increases

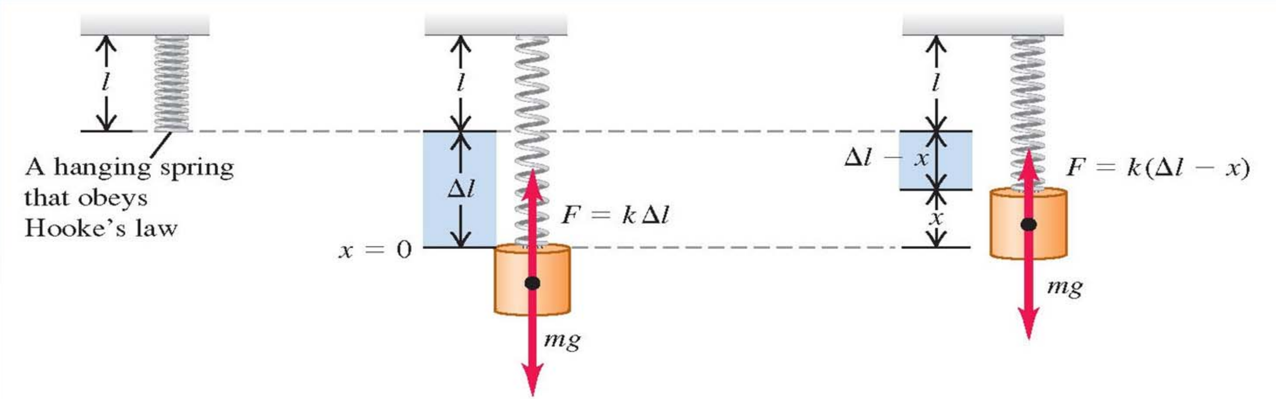
Vertical oscillation



Vertical harmonic oscillation

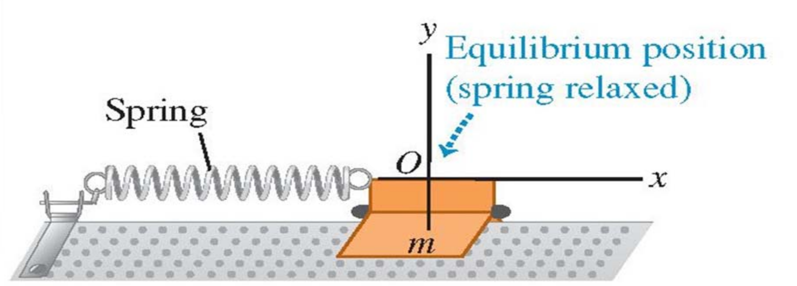
Vertical oscillation

Gravity will stretch the spring to a new eqilibrium position.



Horizontal oscillation

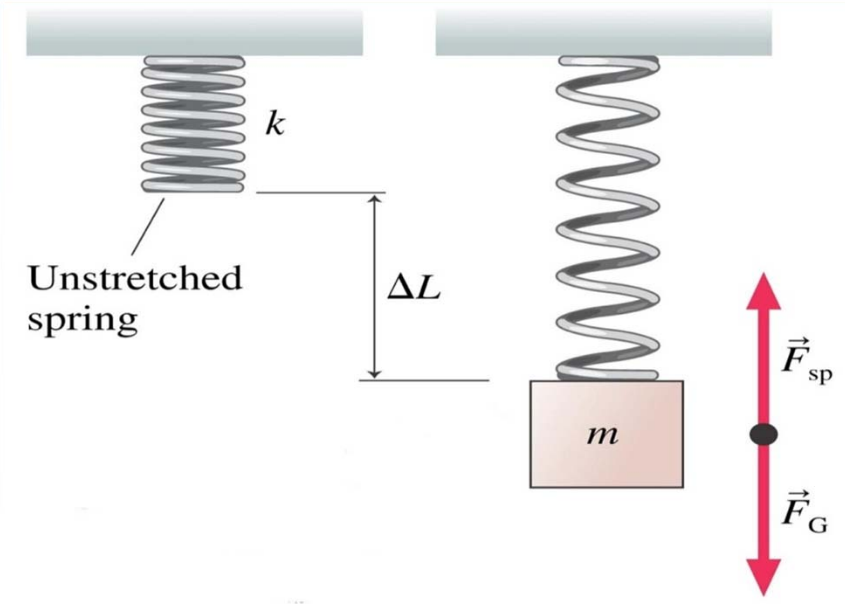
Gravity will not stretch the spring to a new eqilibrium position.



However, the oscillations will be the same.

Vertical harmonic oscillation

Without oscillations: How much is the spring pulled out ?

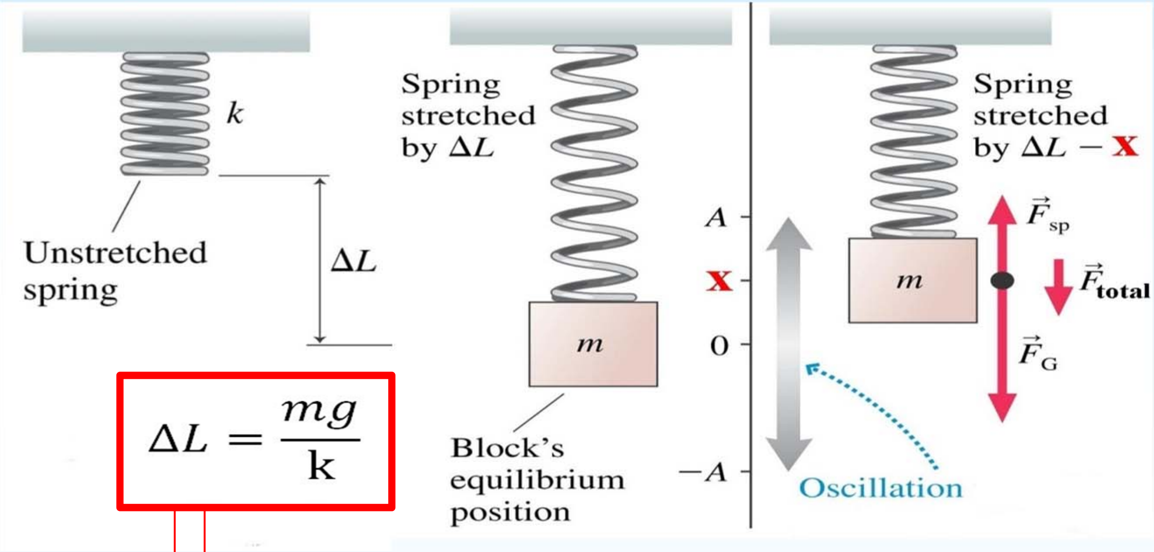


$$\vec{F}_{total} = \vec{F}_{sp} - \vec{F}_G = k\Delta L - mg$$

$$\vec{F}_{total} = m\vec{a} = 0$$

$$\Delta L = \frac{mg}{k}$$

Vertical harmonic oscillation



With oscillations:

Add up the forces !

$$\Delta L = \frac{mg}{k}$$

$$\vec{F}_{total} = \vec{F}_{sp} - \vec{F}_G = k(\Delta L - x) - mg \quad = -kx$$

Vertical harmonic oscillation

Hooke's law:

$$\vec{F}_{total} = \vec{F}_{sp} - \vec{F}_G = -kx$$

Newton's law:

$$\vec{F}_{total} = m\vec{a} \neq 0$$

$$-kx = m\vec{a} = m\frac{\partial^2 x}{\partial t^2} \Rightarrow \frac{\partial^2 x}{\partial t^2} + \frac{k}{m}x = 0$$

This differential equation has the following solution:

$$x = A\cos(\omega t + \varphi) \qquad \omega = \sqrt{\frac{k}{m}}$$

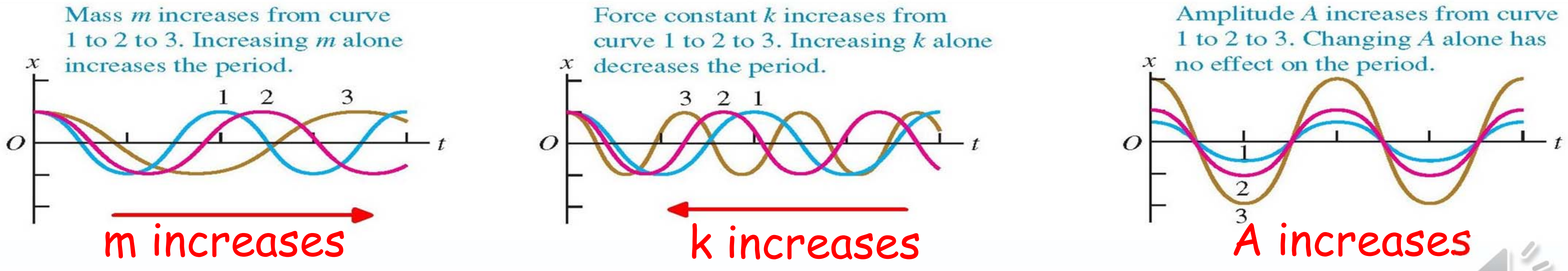
Harmonic oscillation: Forces

$$\omega = \sqrt{\frac{k}{m}}$$

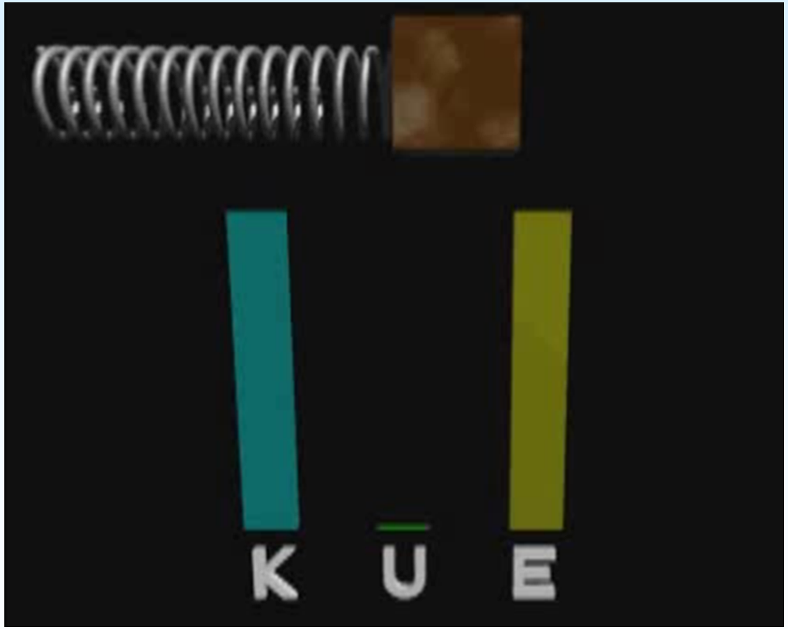
$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \quad \text{(simple harmonic motion)}$$

$$T = \frac{1}{f} = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} \quad \text{(simple harmonic motion)}$$

Note: f and T depend only on k and m.
Not the amplitude !

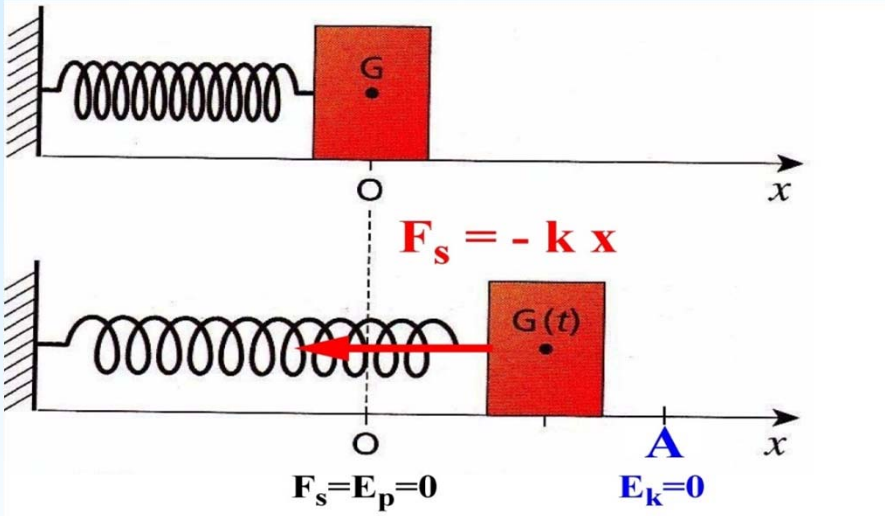


Energy and harmonic oscillations



https://www.youtube.com/watch?v=PL5g_IwrC5U

Harmonic oscillation: Energy

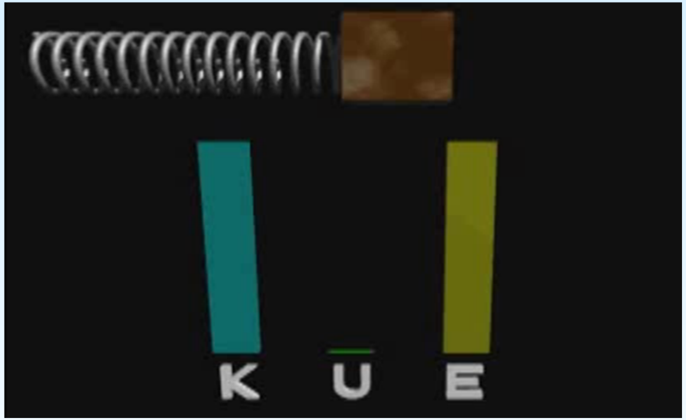


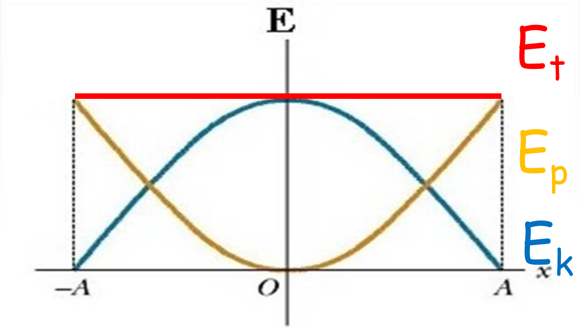
Kinetic energy: $E_k = \frac{mv^2}{2}$ where $v = -\omega A \sin(\omega t)$

Potential energy: $E_p = \frac{kx^2}{2}$ where $x = A\cos(\omega t)$

Total energy: $E_t = E_k + E_p = \frac{kA^2}{2}$ ($E_k = 0$ for $x = A$)

The total mechanical energy is constant





$E_t = \frac{1}{2}kA^2$

E_p

E_k

Harmonic oscillation: Energy

$x = A\cos(\omega t+\phi)$

$v = -\omega A\sin(\omega t+\phi)$

$\omega = \sqrt{\frac{k}{m}}$

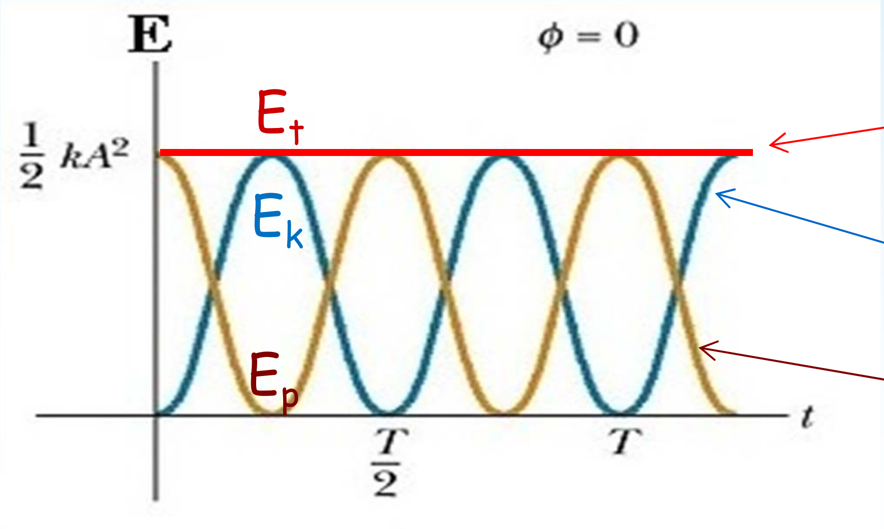
$E_p = \frac{1}{2}kx^2 = \frac{1}{2}kA^2\cos^2(\omega t+\phi)$

$E_k = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2A^2\sin^2(\omega t+\phi) = \frac{1}{2}kA^2\sin^2(\omega t+\phi)$

$E_t = E_p + E_k = \frac{1}{2}kA^2[\cos^2(\omega t+\phi) + \sin^2(\omega t+\phi)] = \frac{1}{2}kA^2$

Harmonic oscillation: Energy

The time dependence of the energy is described by the square of sine and cosine functions:



$$E_t = E_p + E_k = \tfrac{1}{2}kA^2$$

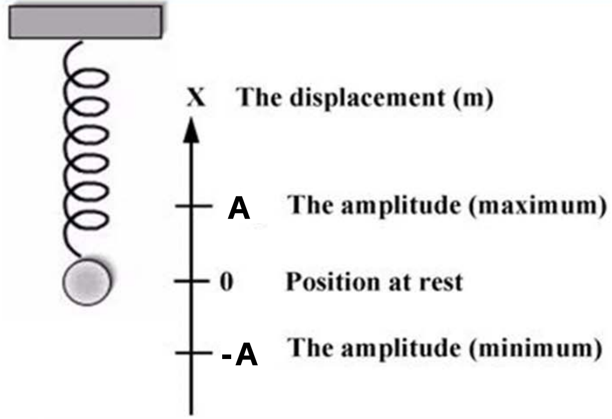
$$E_k = \tfrac{1}{2}mv^2 = \tfrac{1}{2}kA^2\sin^2(\omega t)$$

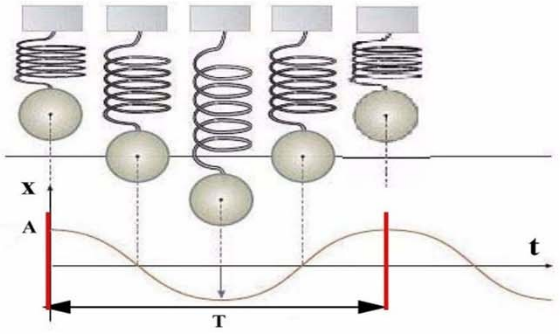
$$E_p = \tfrac{1}{2}kx^2 = \tfrac{1}{2}kA^2\cos^2(\omega t)$$

SUMMARY

Harmonic oscillation

Harmonic oscillation: Summary





$\phi = \mathrm{acos}(x_0 / A) = \mathrm{acos}(A/A) = 0$

- **x** The displacement (m)
- **A** The amplitude (m)
- **t** Time (s)
- **T** Period (s)
- **f** Frequency (Hz) $= 1 / T$
- **ω** Angular Frequency (Hz) $= 2\pi / T = 2\pi f$

$$x = A\cos(\omega t + \phi) \longrightarrow x_{max} = A$$

$$v = \frac{dx}{dt} = -\omega A\sin(\omega t + \phi) \longrightarrow v_{max} = \omega A$$

$$a = \frac{dv}{dt} = -\omega^2 A\cos(\omega t + \phi) \longrightarrow a_{max} = \omega^2 A$$

Harmonic oscillation: Summary

Harmonic oscillations in a spring are described by the equation

$$\frac{\partial^2 x}{\partial t^2} + \frac{k}{m}x = 0$$

if F = -kX

which has the solution

$$x = A\cos(\omega t + \varphi)$$

$$\omega = \sqrt{\frac{k}{m}}$$

Kinetic energy: $E_k = \frac{1}{2}mv^2 = \frac{1}{2}kA^2\sin^2(\omega t)$

Potential energy: $E_p = \frac{1}{2}kx^2 = \frac{1}{2}kA^2\cos^2(\omega t)$

Total energy: $E_t = E_p + E_k = \frac{1}{2}kA^2$