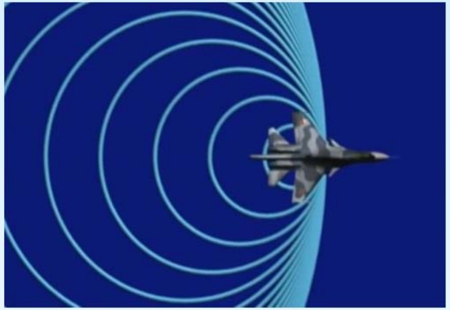
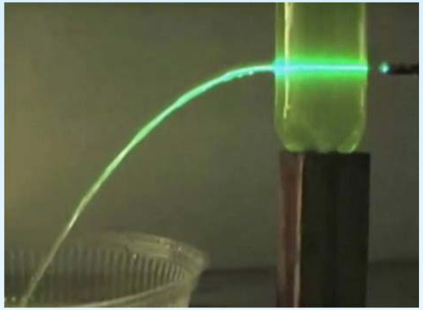
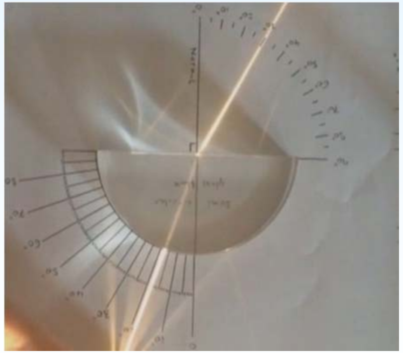
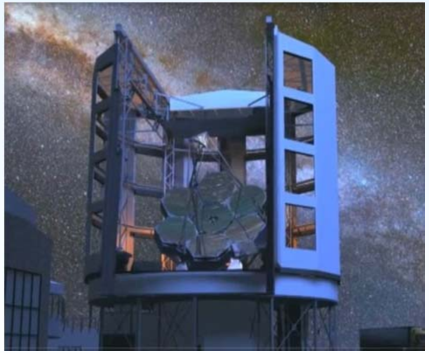
Wavemechanics and optics





Chapter 15 – Mechanical waves

Vincent Hedberg - Lunds Universitet

Content

Part 1. Transverse waves

What is a wave ?

- A wave is when a system is disturbed from its equilibrium and the disturbance is moving.
- A mechanical wave propagates in a medium.
- An electromagnetic wave can propagate without a medium in vacuum.
- Waves transports energy but not matter.

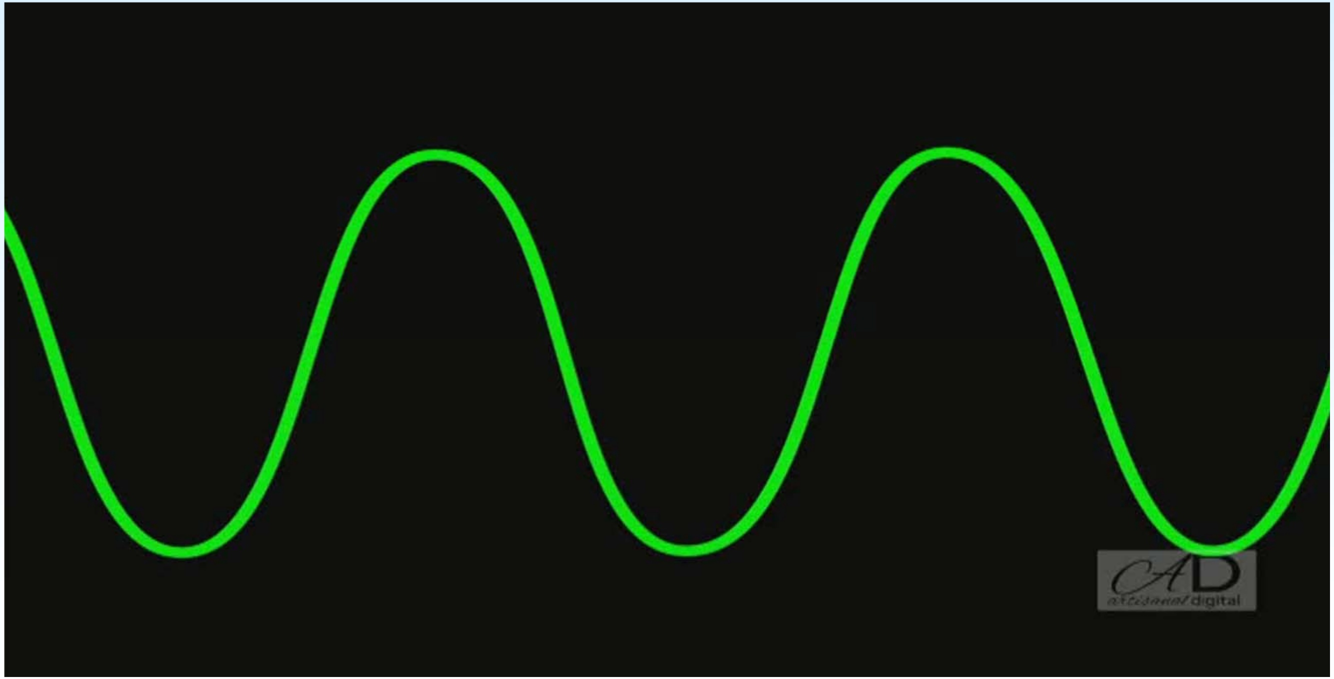
Mechanical waves: Transverse waves

Transverse wave: The medium moves transverse to the wave direction.

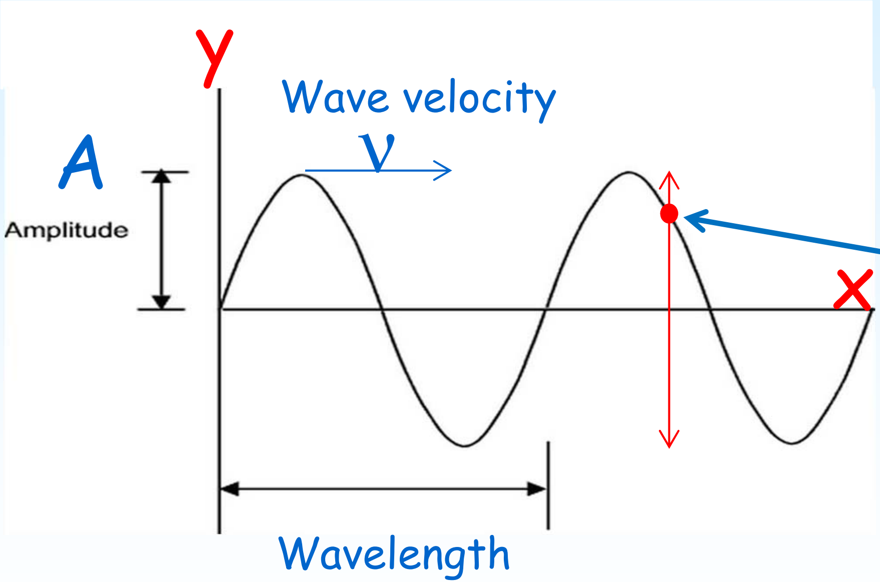
https://www.youtube.com/watch?v=FUBGrH-PbsU

Mechanical waves: Transverse waves

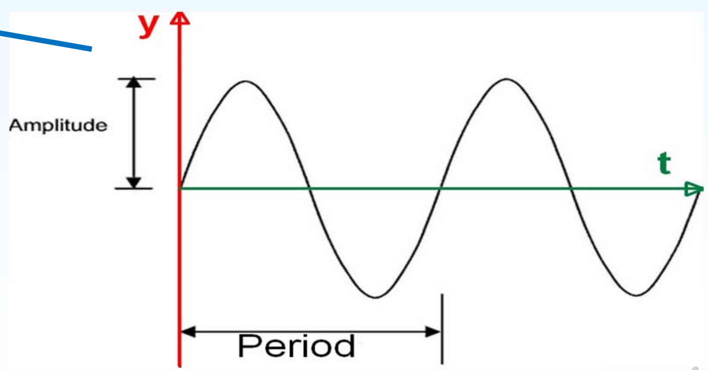
A special transverse wave is the sinusoidal wave:



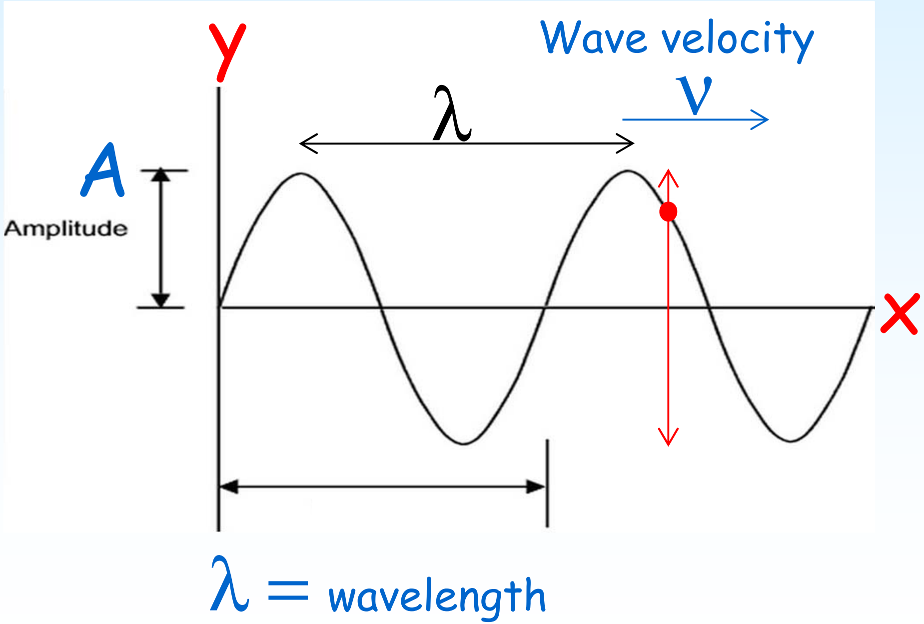
Transverse sinusoidal waves



Every point on the wave moves up and down like an harmonic oscillator with the period T.



Definitions:



A: Amplitude (m)

T: Period (s)

λ: Wavelength (m)

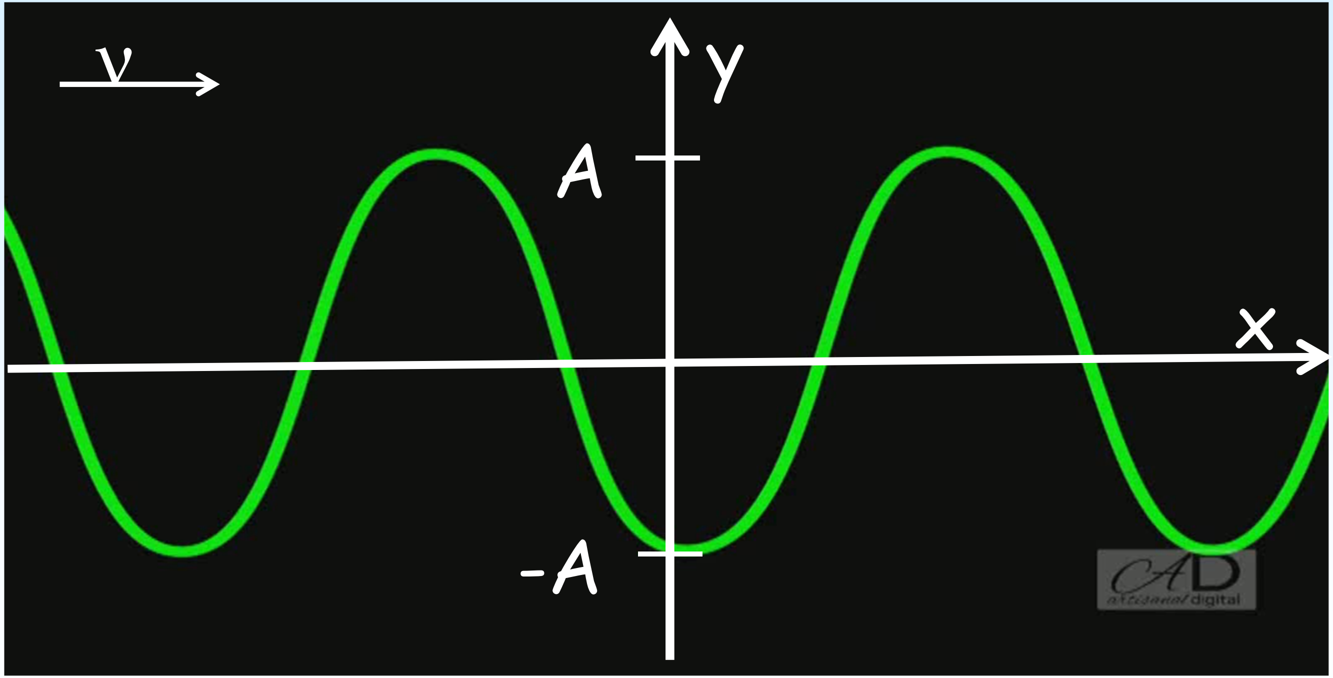
v: Wave speed (m/s) = λ / T

f: Frequency (Hz) = $1 / T$

ω: Angular frequency (radians/s) = $2 \pi f$

k: Wave number (radians/m) = $2 \pi / \lambda$

Mechanical waves: Transverse waves



Part 2. Longitudinal waves

Longitudinal waves created by "The Offspring":

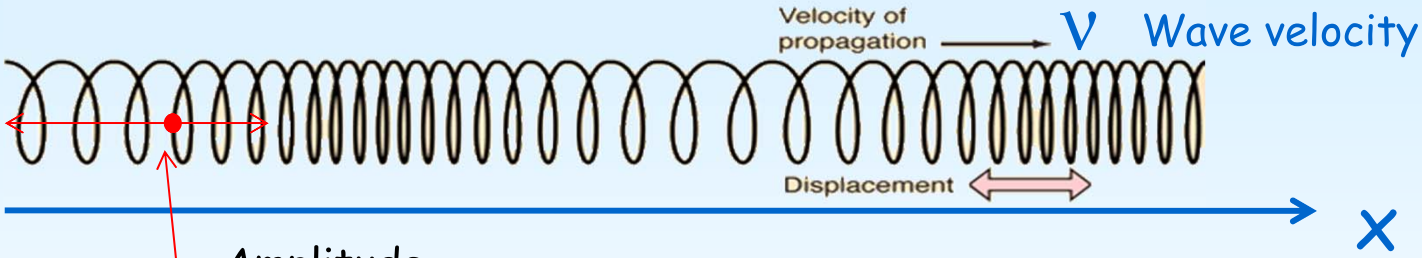
Why don't you get a job ?

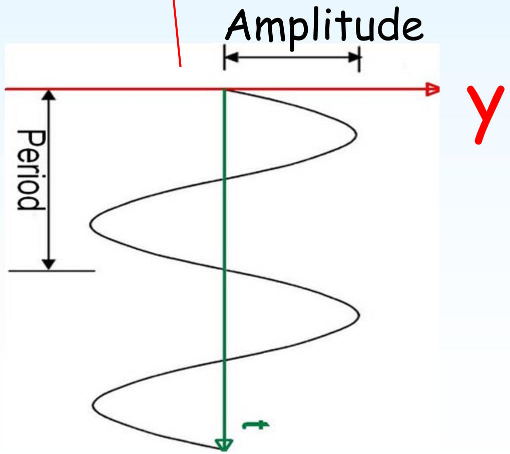
Mechanical waves: Longitudinal waves

Longitudinal waves:

The medium moves in the wave direction.

Longitudinal sinusoidal waves



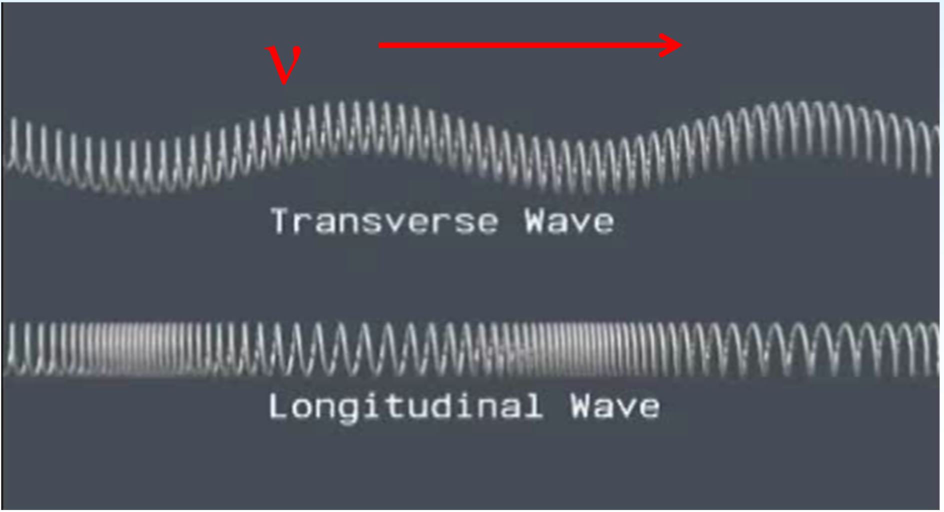


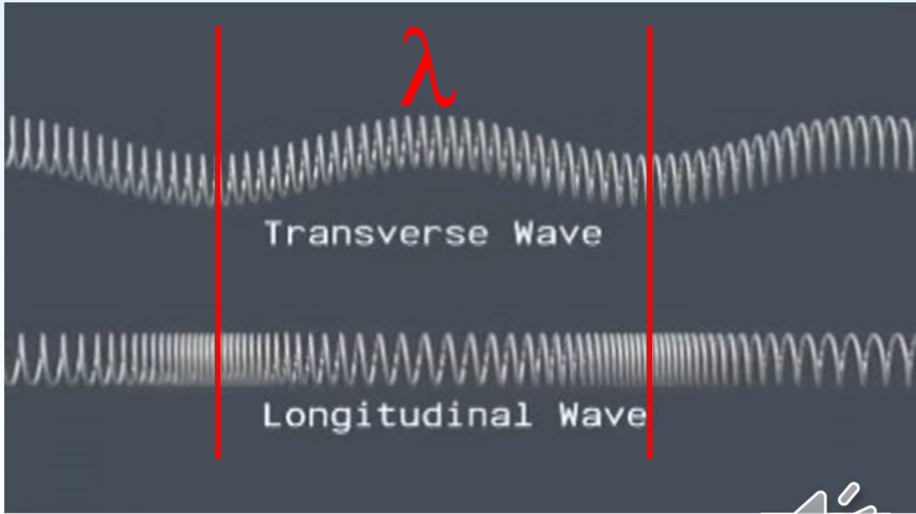
Every point on the wave moves sideways like an harmonic oscillator with the period T.

Mechanical waves: Longitudinal waves

What is the wavelength (λ) for a sinusoidal wave ?
What is the wave velocity (v) ?

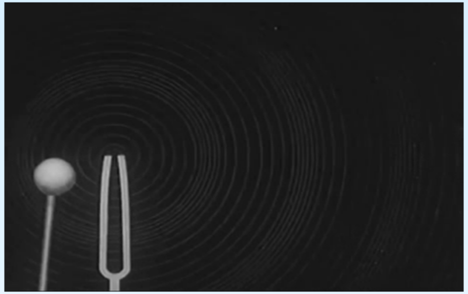
$$v = \lambda / T = \lambda f$$

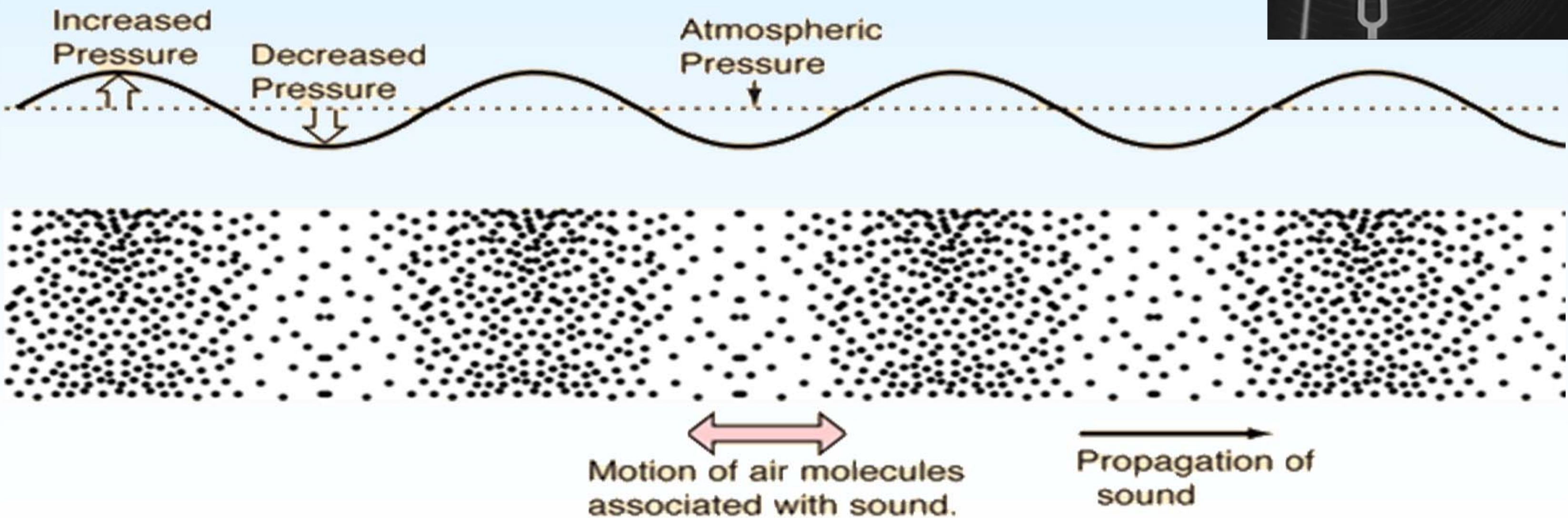




Mechanical waves: Longitudinal waves

Sound = longitudinal waves in air.





Part 3. Problems

$$\frac{1}{n}\sin x = ?$$

$$\frac{1}{\cancel{n}}\mathrm{si}\cancel{n}\, x =$$

$$six = 6$$

Mechanical waves: Problems

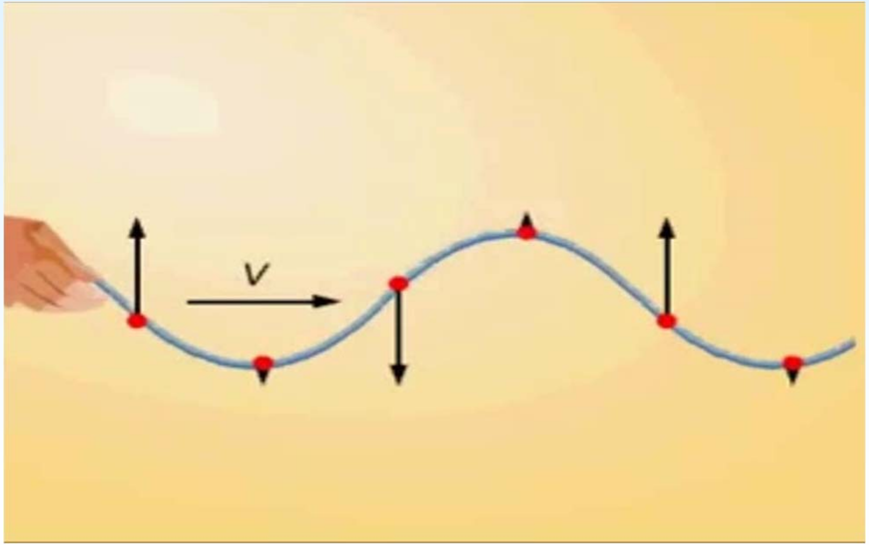
The speed of sound depends on the temperature and is
344 m/s at 20 degrees.

What, then, is the wavelength of sound with the frequency 262 Hz ?

$$v = f \lambda$$

$$\lambda = \frac{v}{f} = \frac{344 \text{ m/s}}{262 \text{ Hz}} = \frac{344 \text{ m/s}}{262 \text{ s}^{-1}} = 1.31 \text{ m}$$

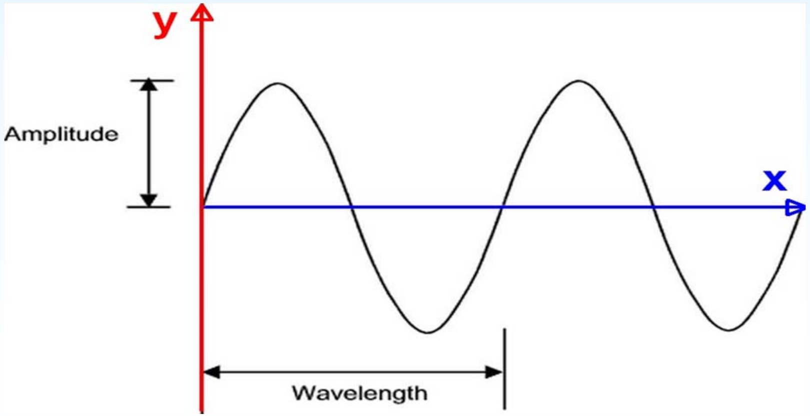
Part 4. The wavefunction



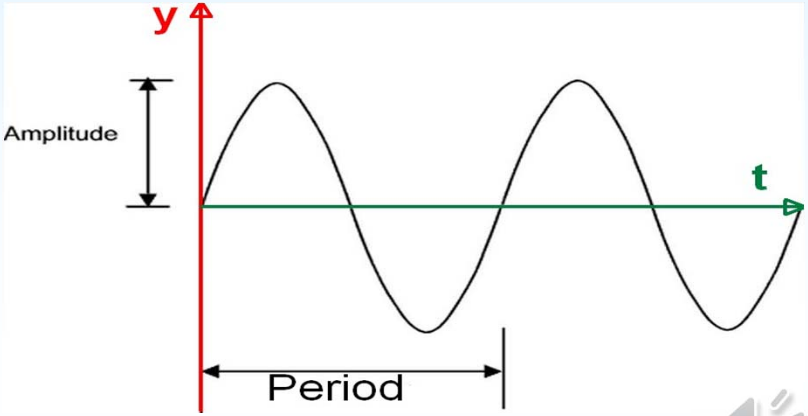
Mechanical waves: The wavefunction

The wavefunction y(x,t): The wave function describes the height of the wave as a function of both distance and time.

The height of the wave as a function of distance x:

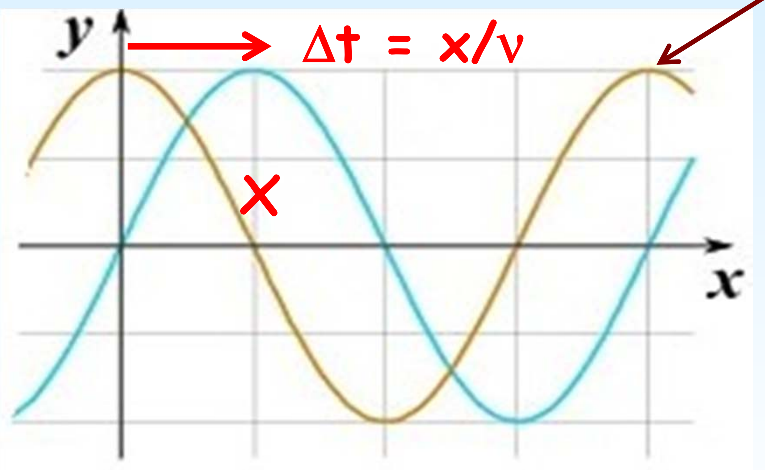


The height of the wave as a function of time t:



Mechanical waves: The wavefunction

A wave travels the distance x over the time Δt:



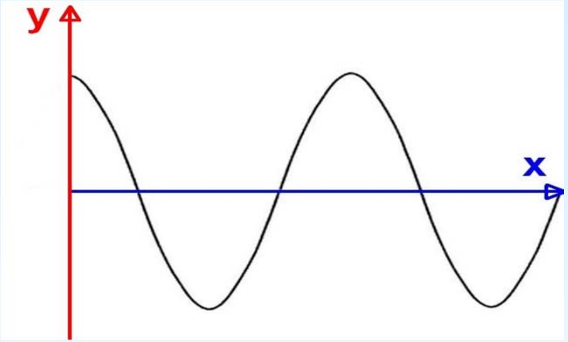
Suppose that the point at x = 0 can be described by $y(0,t) = A\cos(\omega t)$

Another point at distance x will have the same y as the wave had at a time $\Delta t = x/v$ earlier e.g. substitute t with $t-x/v$:

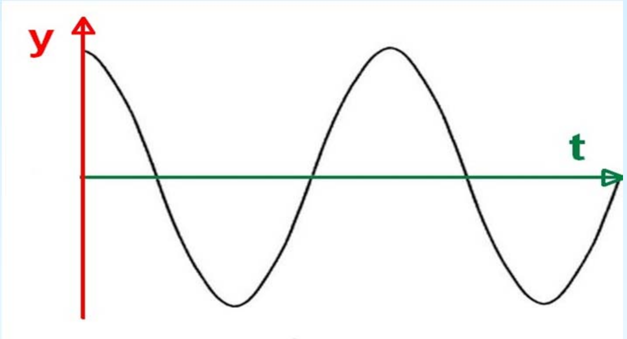
$$y(x,t) = A\cos(\omega(t-x/v))$$
$$y(x,t) = A\cos(\omega(x/v-t))$$ because $\cos(-x) = \cos(x)$

$$y(x,t) = A\cos(\omega(x/v-t)) = A\cos(2\pi f(x/v-t)) = A\cos(2\pi(fx/v-ft))$$
$$y(x,t) = A\cos(2\pi(x/\lambda-t/T)) = A\cos(kx-\omega t)$$

$f/v = 1/\lambda$ $f=1/T$



$$y(x, t = 0) = A\cos kx$$



$$y(x = 0, t) = A\cos \omega t$$

$$y(x, t) = A\cos(kx - \omega t)$$ (sinusoidal wave moving in $+x$-direction)

+ if moving in the negative x direction

Mechanical waves: The wavefunction

$$y(x, t) = A\cos(kx - \omega t)$$ (sinusoidal wave moving in $+x$-direction)

Amplitude: A

Wavenumber: $k = \dfrac{2\pi}{\lambda}$

$v = \lambda / T$
$f = 1 / T$

Angular frequency: $\omega = \dfrac{2\pi}{T}$

$$v = \lambda / T = (2\pi/k) / (2\pi/\omega) = \omega / k$$

The wavefunction:

$$y(x,t) = A\cos(kx - \omega t)$$

Velocity and acceleration:

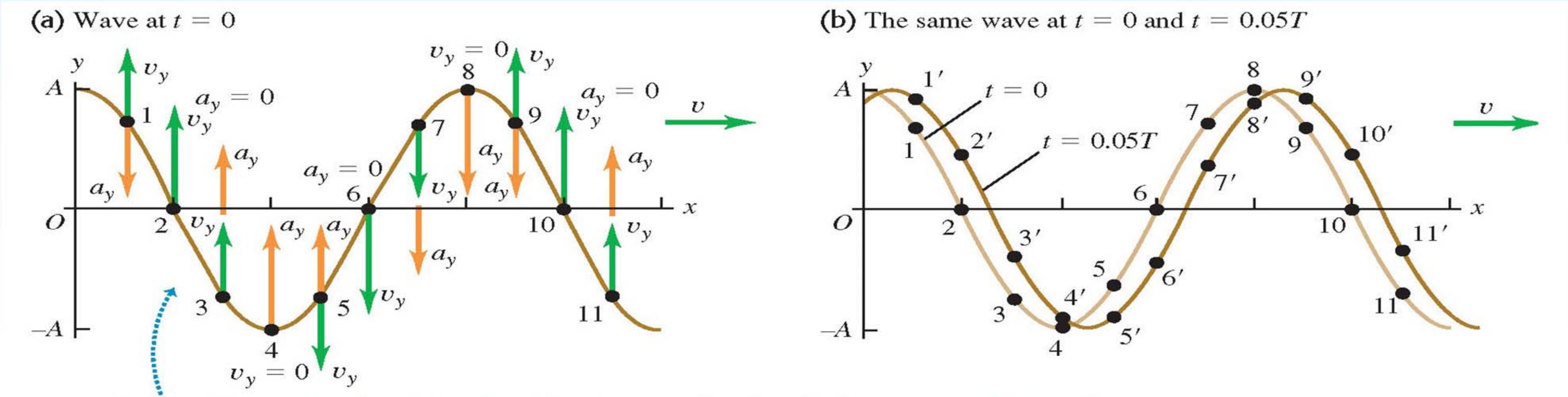
$$v_y(x,t) = \frac{\partial y(x,t)}{\partial t} = \omega A \sin(kx - \omega t)$$

$$a_y(x,t) = \frac{\partial^2 y(x,t)}{\partial t^2} = -\omega^2 A\cos(kx - \omega t) = -\omega^2 y(x,t)$$

Mechanical waves: The wavefunction

Velocity: $$v_y(x,t) = \frac{\partial y(x,t)}{\partial t} = \omega A \sin(kx - \omega t)$$

Acceleration: $$a_y(x,t) = \frac{\partial^2 y(x,t)}{\partial t^2} = -\omega^2 A \cos(kx - \omega t) = -\omega^2 y(x,t)$$



- Acceleration a_y at each point on the string is proportional to displacement y at that point.
- Acceleration is upward where string curves upward, downward where string curves downward.

The equation for the standard model in particle physics:

$$
\begin{gathered}
\mathcal{L}_{SM} = -\tfrac{1}{2}\partial_\nu g^a_\mu \partial_\nu g^a_\mu - g_s f^{abc}\partial_\mu g^a_\nu g^b_\mu g^c_\nu - \tfrac{1}{4}g_s^2 f^{abc}f^{ade}g^b_\mu g^c_\nu g^d_\mu g^e_\nu - \partial_\nu W^+_\mu \partial_\nu W^-_\mu - \\
M^2 W^+_\mu W^-_\mu - \tfrac{1}{2}\partial_\nu Z^0_\mu \partial_\nu Z^0_\mu - \tfrac{1}{2c_w^2}M^2 Z^0_\mu Z^0_\mu - \tfrac{1}{2}\partial_\mu A_\nu \partial_\mu A_\nu - igc_w(\partial_\nu Z^0_\mu(W^+_\mu W^-_\nu - W^+_\nu W^-_\mu) - \\
Z^0_\nu(W^+_\mu \partial_\nu W^-_\mu - W^-_\mu \partial_\nu W^+_\mu) + Z^0_\mu(W^+_\nu \partial_\nu W^-_\mu - W^-_\nu \partial_\nu W^+_\mu)) - igs_w(\partial_\nu A_\mu(W^+_\mu W^-_\nu - \\
W^+_\nu W^-_\mu) - A_\nu(W^+_\mu \partial_\nu W^-_\mu - W^-_\mu \partial_\nu W^+_\mu) + A_\mu(W^+_\nu \partial_\nu W^-_\mu - W^-_\nu \partial_\nu W^+_\mu)) - \\
\tfrac{1}{2}g^2 W^+_\mu W^-_\mu W^+_\nu W^-_\nu + \tfrac{1}{2}g^2 W^+_\mu W^-_\nu W^+_\mu W^-_\nu + g^2 c_w^2(Z^0_\mu W^+_\mu Z^0_\nu W^-_\nu - Z^0_\mu Z^0_\mu W^+_\nu W^-_\nu) + \\
g^2 s_w^2(A_\mu W^+_\mu A_\nu W^-_\nu - A_\mu A_\mu W^+_\nu W^-_\nu) + g^2 s_w c_w(A_\mu Z^0_\nu(W^+_\mu W^-_\nu - W^+_\nu W^-_\mu) - \\
2A_\mu Z^0_\mu W^+_\nu W^-_\nu) - \tfrac{1}{2}\partial_\mu H \partial_\mu H - 2M^2\alpha_h H^2 - \partial_\mu \phi^+ \partial_\mu \phi^- - \tfrac{1}{2}\partial_\mu \phi^0 \partial_\mu \phi^0 - \\
\beta_h\left(\tfrac{2M^2}{g^2} + \tfrac{2M}{g}H + \tfrac{1}{2}(H^2 + \phi^0\phi^0 + 2\phi^+\phi^-)\right) + \tfrac{2M^4}{g^2}\alpha_h - g\alpha_h M\left(H^3 + H\phi^0\phi^0 + 2H\phi^+\phi^-\right) - \\
\tfrac{1}{8}g^2\alpha_h\left(H^4 + (\phi^0)^4 + 4(\phi^+\phi^-)^2 + 4(\phi^0)^2\phi^+\phi^- + 4H^2\phi^+\phi^- + 2(\phi^0)^2H^2\right) - gMW^+_\mu W^-_\mu H - \\
\tfrac{1}{2}g\tfrac{M}{c_w^2}Z^0_\mu Z^0_\mu H - \tfrac{1}{2}ig\left(W^+_\mu(\phi^0\partial_\mu\phi^- - \phi^-\partial_\mu\phi^0) - W^-_\mu(\phi^0\partial_\mu\phi^+ - \phi^+\partial_\mu\phi^0)\right) + \\
\tfrac{1}{2}g\left(W^+_\mu(H\partial_\mu\phi^- - \phi^-\partial_\mu H) + W^-_\mu(H\partial_\mu\phi^+ - \phi^+\partial_\mu H)\right) + \tfrac{1}{2}g\tfrac{1}{c_w}(Z^0_\mu(H\partial_\mu\phi^0 - \phi^0\partial_\mu H) + \\
M\left(\tfrac{1}{c_w}Z^0_\mu\partial_\mu\phi^0 + W^+_\mu\partial_\mu\phi^- + W^-_\mu\partial_\mu\phi^+\right) - ig\tfrac{s_w^2}{c_w}MZ^0_\mu(W^+_\mu\phi^- - W^-_\mu\phi^+) + igs_w MA_\mu(W^+_\mu\phi^- - \\
W^-_\mu\phi^+) - ig\tfrac{1-2c_w^2}{2c_w}Z^0_\mu(\phi^+\partial_\mu\phi^- - \phi^-\partial_\mu\phi^+) + igs_w A_\mu(\phi^+\partial_\mu\phi^- - \phi^-\partial_\mu\phi^+) - \\
\tfrac{1}{4}g^2 W^+_\mu W^-_\mu\left(H^2 + (\phi^0)^2 + 2\phi^+\phi^-\right) - \tfrac{1}{8}g^2\tfrac{1}{c_w^2}Z^0_\mu Z^0_\mu\left(H^2 + (\phi^0)^2 + 2(2s_w^2 - 1)^2\phi^+\phi^-\right) - \\
\tfrac{1}{2}g^2\tfrac{s_w^2}{c_w}Z^0_\mu\phi^0(W^+_\mu\phi^- + W^-_\mu\phi^+) - \tfrac{1}{2}ig^2\tfrac{s_w^2}{c_w}Z^0_\mu H(W^+_\mu\phi^- - W^-_\mu\phi^+) + \tfrac{1}{2}g^2 s_w A_\mu\phi^0(W^+_\mu\phi^- + \\
W^-_\mu\phi^+) + \tfrac{1}{2}ig^2 s_w A_\mu H(W^+_\mu\phi^- - W^-_\mu\phi^+) - g^2\tfrac{s_w}{c_w}(2c_w^2 - 1)Z^0_\mu A_\mu\phi^+\phi^- - g^2 s_w^2 A_\mu A_\mu\phi^+\phi^- + \\
\tfrac{1}{2}ig_s\lambda^a_{ij}(\bar{q}^\sigma_i\gamma^\mu q^\sigma_j)g^a_\mu - \bar{e}^\lambda(\gamma\partial + m^\lambda_e)e^\lambda - \bar{\nu}^\lambda(\gamma\partial + m^\lambda_\nu)\nu^\lambda - \bar{u}^\lambda_j(\gamma\partial + m^\lambda_u)u^\lambda_j - \bar{d}^\lambda_j(\gamma\partial + m^\lambda_d)d^\lambda_j + \\
igs_w A_\mu\left(-(\bar{e}^\lambda\gamma^\mu e^\lambda) + \tfrac{2}{3}(\bar{u}^\lambda_j\gamma^\mu u^\lambda_j) - \tfrac{1}{3}(\bar{d}^\lambda_j\gamma^\mu d^\lambda_j)\right) + \tfrac{ig}{4c_w}Z^0_\mu\{(\bar{\nu}^\lambda\gamma^\mu(1+\gamma^5)\nu^\lambda) + (\bar{e}^\lambda\gamma^\mu(4s_w^2 - \\
1 - \gamma^5)e^\lambda) + (\bar{d}^\lambda_j\gamma^\mu(\tfrac{4}{3}s_w^2 - 1 - \gamma^5)d^\lambda_j) + (\bar{u}^\lambda_j\gamma^\mu(1 - \tfrac{8}{3}s_w^2 + \gamma^5)u^\lambda_j)\} + \\
\tfrac{ig}{2\sqrt{2}}W^+_\mu\left((\bar{\nu}^\lambda\gamma^\mu(1+\gamma^5)U^{lep}{}_{\lambda\kappa}e^\kappa) + (\bar{u}^\lambda_j\gamma^\mu(1+\gamma^5)C_{\lambda\kappa}d^\kappa_j)\right) + \\
\tfrac{ig}{2\sqrt{2}}W^-_\mu\left((\bar{e}^\kappa U^{lep\dagger}{}_{\kappa\lambda}\gamma^\mu(1+\gamma^5)\nu^\lambda) + (\bar{d}^\kappa_j C^\dagger_{\kappa\lambda}\gamma^\mu(1+\gamma^5)u^\lambda_j)\right) + \\
\tfrac{ig}{2M\sqrt{2}}\phi^+\left(-m^\kappa_e(\bar{\nu}^\lambda U^{lep}{}_{\lambda\kappa}(1-\gamma^5)e^\kappa) + m^\lambda_\nu(\bar{\nu}^\lambda U^{lep}{}_{\lambda\kappa}(1+\gamma^5)e^\kappa\right) + \\
\tfrac{ig}{2M\sqrt{2}}\phi^-\left(m^\lambda_e(\bar{e}^\lambda U^{lep\dagger}{}_{\lambda\kappa}(1+\gamma^5)\nu^\kappa) - m^\kappa_\nu(\bar{e}^\lambda U^{lep\dagger}{}_{\lambda\kappa}(1-\gamma^5)\nu^\kappa\right) - \tfrac{g}{2}\tfrac{m^\lambda_\nu}{M}H(\bar{\nu}^\lambda\nu^\lambda) - \\
\tfrac{g}{2}\tfrac{m^\lambda_e}{M}H(\bar{e}^\lambda e^\lambda) + \tfrac{ig}{2}\tfrac{m^\lambda_\nu}{M}\phi^0(\bar{\nu}^\lambda\gamma^5\nu^\lambda) - \tfrac{ig}{2}\tfrac{m^\lambda_e}{M}\phi^0(\bar{e}^\lambda\gamma^5 e^\lambda) - \tfrac{1}{4}\bar{\nu}_\lambda M^R_{\lambda\kappa}(1-\gamma_5)\hat{\nu}_\kappa - \\
\tfrac{1}{4}\overline{\bar{\nu}_\lambda M^R_{\lambda\kappa}(1-\gamma_5)\hat{\nu}_\kappa} + \tfrac{ig}{2M\sqrt{2}}\phi^+\left(-m^\kappa_d(\bar{u}^\lambda_j C_{\lambda\kappa}(1-\gamma^5)d^\kappa_j) + m^\lambda_u(\bar{u}^\lambda_j C_{\lambda\kappa}(1+\gamma^5)d^\kappa_j\right) + \\
\tfrac{ig}{2M\sqrt{2}}\phi^-\left(m^\lambda_d(\bar{d}^\lambda_j C^\dagger_{\lambda\kappa}(1+\gamma^5)u^\kappa_j) - m^\kappa_u(\bar{d}^\lambda_j C^\dagger_{\lambda\kappa}(1-\gamma^5)u^\kappa_j\right) - \tfrac{g}{2}\tfrac{m^\lambda_u}{M}H(\bar{u}^\lambda_j u^\lambda_j) - \tfrac{g}{2}\tfrac{m^\lambda_d}{M}H(\bar{d}^\lambda_j d^\lambda_j) + \\
\tfrac{ig}{2}\tfrac{m^\lambda_u}{M}\phi^0(\bar{u}^\lambda_j\gamma^5 u^\lambda_j) - \tfrac{ig}{2}\tfrac{m^\lambda_d}{M}\phi^0(\bar{d}^\lambda_j\gamma^5 d^\lambda_j)
\end{gathered}
$$

Part 5. The wave equation

Mechanical waves: The wave equation

The wavefunction: $y(x, t) = A\cos(kx - \omega t)$

Velocity and acceleration:

$$v_y(x, t) = \frac{\partial y(x, t)}{\partial t} = \omega A \sin(kx - \omega t)$$

$$a_y(x, t) = \frac{\partial^2 y(x, t)}{\partial t^2} = -\omega^2 A\cos(kx - \omega t) = -\omega^2 y(x, t)$$

Curvature:

$$\frac{\partial^2 y(x, t)}{\partial x^2} = -k^2 A\cos(kx - \omega t) = -k^2 y(x, t)$$

$$\frac{\partial^2 y(x, t)/\partial t^2}{\partial^2 y(x, t)/\partial x^2} = \frac{\omega^2}{k^2} = v^2$$

$v = \lambda / T = \omega / k$

The wave equation:

$$\frac{\partial^2 y(x, t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y(x, t)}{\partial t^2}$$

Mechanical waves: The wave equation

The wave equation: $$\frac{\partial^2 y(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y(x,t)}{\partial t^2}$$

The wave equation describes also waves that are not sinusoidal !

It even describes waves that are not periodic !

And waves in three dimensions !

Part 6. Problems

$$\frac{1}{n}\sin x = ?$$

$$\frac{1}{\cancel{n}}\mathrm{si}\cancel{\mathrm{n}}\, x =$$

$$six = 6$$

Mechanical waves: Problems

You wave a rope up and down and create a sine wave with the frequency 2.00 Hz, the amplitude 0.075 m and the wave speed 12.0 m/s.

Calculate the period, the wavelength, the angular frequency and the wave number !

$f = 1/T$
$\omega = 2\pi f$
$v = f\lambda$
$k = 2\pi/\lambda$

Given:

A: Amplitude = 0.075 m

f: Frequency = 1 / T = 2.00 Hz

v: Wave speed = λ / T = 12.0 m/s

To calculate:

T: Period = 1 / f = 0.5 s

λ: Wavelength = v T = 6.00 m

ω: Angular frequency = 2 π f = 4π rad/s

k: Wave number = 2 π / λ = $\frac{1}{3}\pi$ rad/m

Mechanical waves: Problems

At $t = 0$, the rope you hold in your hand ($x = 0$) is in its highest position (0.075 m).

What is the wave function for the oscillations ?

What will be the wave function at $x = 0$ and $x = 3.00$ m ?

Calculated previously:

ω: Angular frequency $= 2\pi f = 4\pi$ rad/s

k: Wave number $= 2\pi/\lambda = \frac{1}{3}\pi$ rad/m

$$y(x,t) = A\cos(kx - \omega t) = 0.075\cos(\tfrac{1}{3}\pi x - 4\pi t)$$

$$y(0,t) = 0.075\cos(-4\pi t) = 0.075\cos(4\pi t)$$

$\cos(-x) = \cos(x)$

$$y(3,t) = 0.075\cos(\pi - 4\pi t) = -0.075\cos(-4\pi t) = -0.075\cos(4\pi t)$$

$\cos(\pi - x) = -\cos(x)$

Part 7. Wave speed and string properties

https://www.youtube.com/watch?v=ttgLyWFINJI

Mechanical waves: Wave speed

Goal:

Find out how the wave speed depends on the properties of the string !

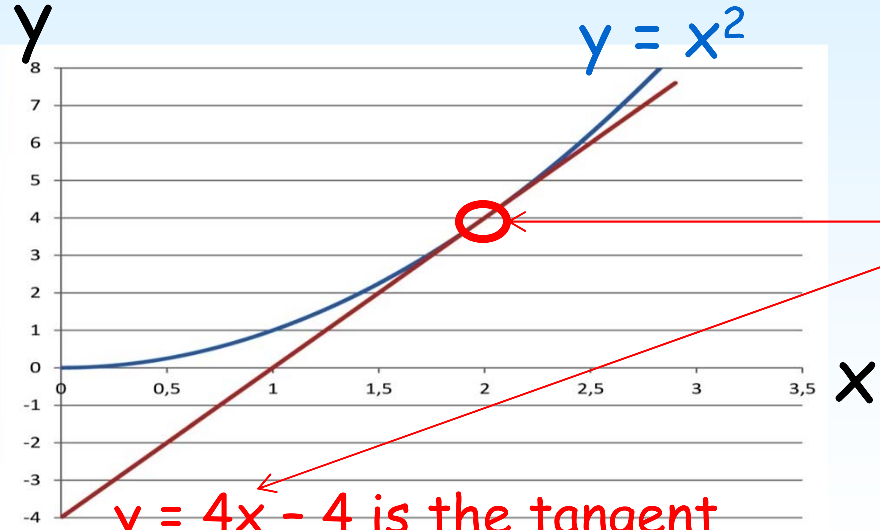
How:

Look at the forces on a small string segment and apply Newton's law:

$F = m\,a$

Mechanical waves: Wave speed

First, a little repetition of the meaning of the derivative.
Using the function $y = x^2$ as an example!



y = 4x – 4 is the tangent to $y=x^2$ in the point x =2

$$y = x^2$$

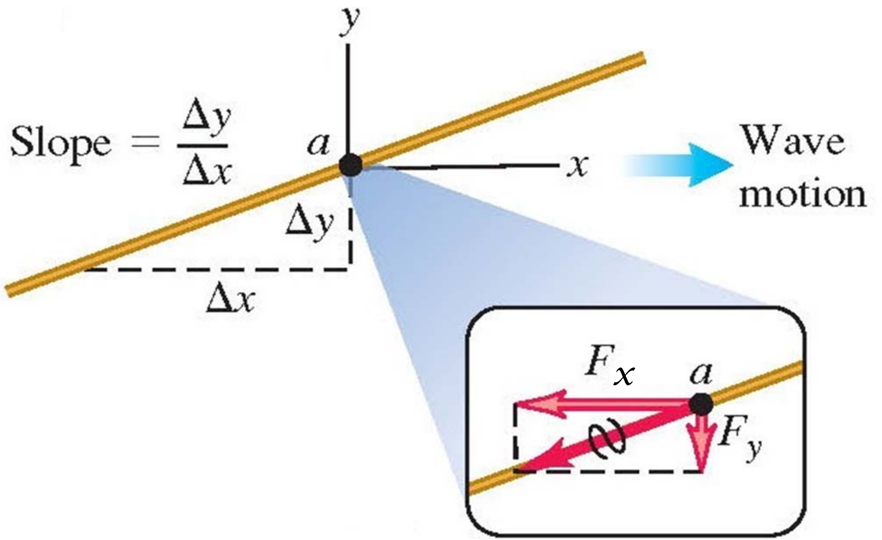
$$\frac{dy}{dx} = 2x$$

$$\frac{dy}{dx} = 4 \quad for\ x = 2$$

The derivation gives the slope of the tangent.

Mechanical waves: Wave speed

We start by looking at the forces at a point on the string !



The ratio of the force in the y-direction to the force in the x-direction is given by the slope of the string which is given by the derivative:

$$Slope = \frac{\Delta y}{\Delta x} = \frac{F_y}{F_x} = \frac{dy}{dx}$$

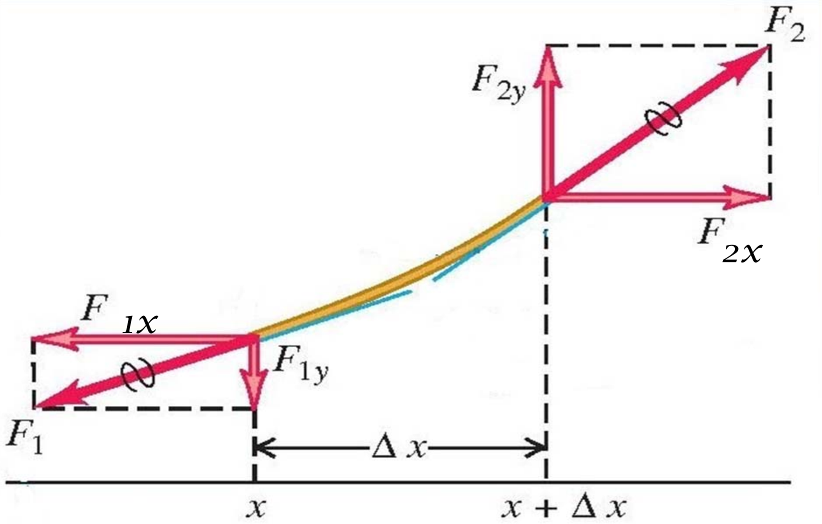
$$F_y(x,t) = -F_x \frac{\partial y(x,t)}{\partial x}$$

F_y is in the negative y direction

Mechanical waves: Wave speed

When the string is at rest, there is only one force in the x direction: The string tension (F).

Then we look at the forces in a segment of the string:

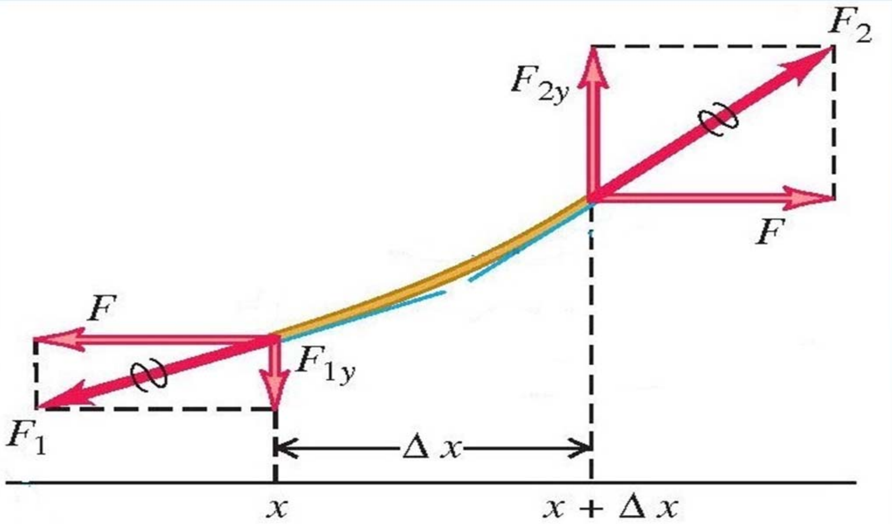


When a transverse wave passes the string it will move up and down but not laterally, i.e., the force in the x-direction will still be = the string tension:

$$F_{1x} = -F_{2x} = F$$

Mechanical waves: Wave speed

Now it's time to use the derivative at the endpoints:



$$\frac{F_{1y}}{F} = -\left(\frac{\partial y}{\partial x}\right)_x$$

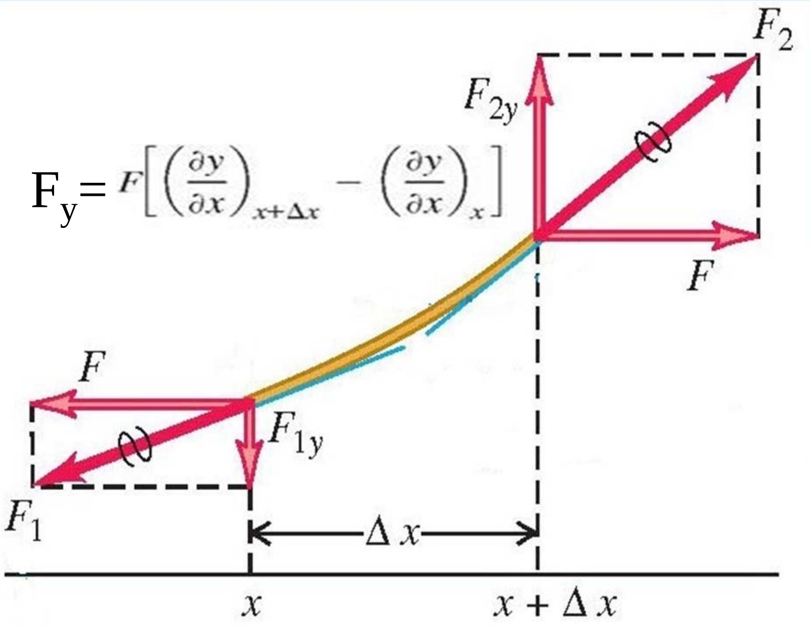
$$\frac{F_{2y}}{F} = \left(\frac{\partial y}{\partial x}\right)_{x+\Delta x}$$

The total force in the y-direction then becomes

$$F_y = F_{1y} + F_{2y} = F\left[\left(\frac{\partial y}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial y}{\partial x}\right)_x\right]$$

Mechanical waves: Wave speed

Now it's time to use Newton's law:



μ: String mass per unit length.

$m = \mu \Delta x$ is the mass of the string segment.

F = ma (Newton's law) where the acceleration is the second derivative with respect to time:

$$F_y = ma = \mu \, \Delta x \frac{\partial^2 y}{\partial t^2}$$

But we have earlier shown that:

$$F_y = F\left[\left(\frac{\partial y}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial y}{\partial x}\right)_x\right]$$

$$F\left[\left(\frac{\partial y}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial y}{\partial x}\right)_x\right] = \mu \, \Delta x \frac{\partial^2 y}{\partial t^2}$$

Mechanical waves: Wave speed

Our new equation can be re-written as:

$$F\left[\left(\frac{\partial y}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial y}{\partial x}\right)_{x}\right] = \mu\,\Delta x \frac{\partial^2 y}{\partial t^2}$$

Divide by $F\Delta x$

$$\frac{\left(\frac{\partial y}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial y}{\partial x}\right)_{x}}{\Delta x} = \frac{\mu}{F}\frac{\partial^2 y}{\partial t^2}$$

When Δx goes to zero this is equivalent to the second derivative with respect to x.

$$\frac{\partial^2 y}{\partial x^2} = \frac{\mu}{F}\frac{\partial^2 y}{\partial t^2}$$

+

You can now combine this equation with the wave equation:

$$\frac{\partial^2 y(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y(x,t)}{\partial t^2}$$

=

The result is that the wave speed becomes:

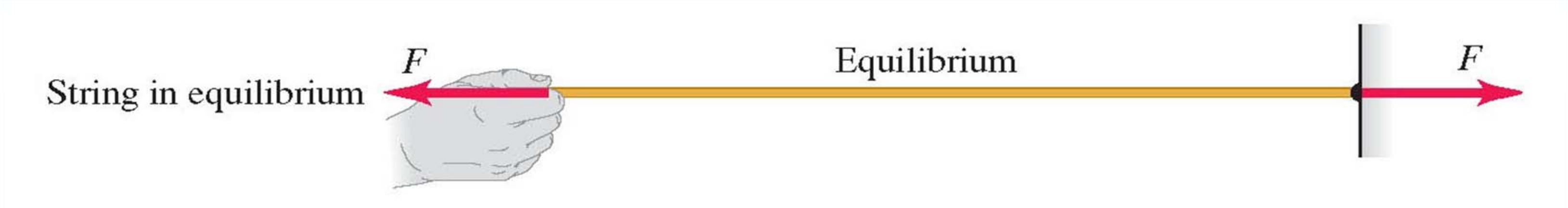
$$v = \sqrt{\frac{F}{\mu}}$$

Mechanical waves: Wave speed

Conclusion: The wave speed depends on two things:

$$v = \sqrt{\frac{F}{\mu}}$$

- F ← The string tension
- μ ← The mass of the string per unit length



More generally:

$$v = \sqrt{\frac{\text{Restoring force returning the system to equilibrium}}{\text{Inertia resisting the return to equilibrium}}}$$

Part 8. Problems

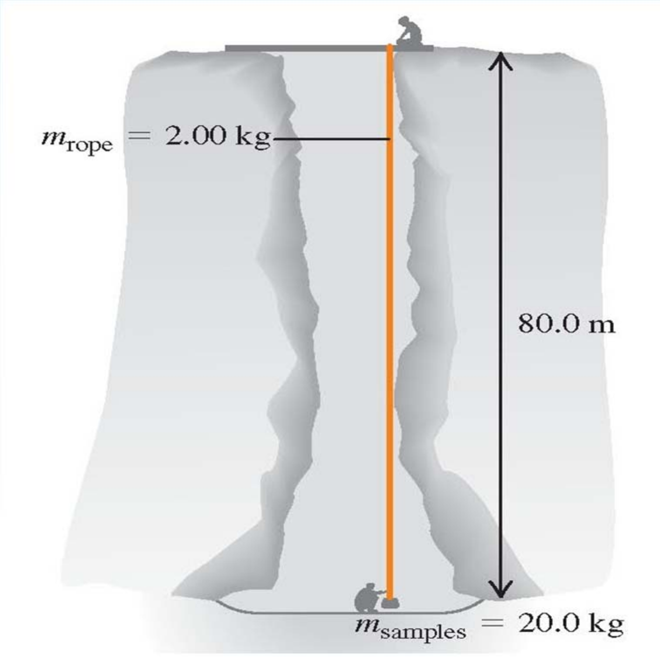
$$\frac{1}{n}\sin x = ?$$

$$\frac{1}{\cancel{n}}\mathrm{si}\cancel{n}\, x =$$

$$six = 6$$

Mechanical waves: Problems

A man in a hole sends a signal by making a wave on a rope at whose end it hangs a weight of 20 kg. What is the speed of the wave in the rope? If the rope is put into sinus oscillation with f = 2Hz, how many wavelengths can fit on the rope ?



The tension in the rope due to the box is

$$F = m_{\text{box}}g = (20.0\ \text{kg})(9.80\ \text{m/s}^2) = 196\ \text{N}$$

and the rope's linear mass density is

$$\mu = \frac{m_{\text{rope}}}{L} = \frac{2.00\ \text{kg}}{80.0\ \text{m}} = 0.0250\ \text{kg/m}$$

the wave speed is

$$v = \sqrt{\frac{F}{\mu}} = \sqrt{\frac{196\ \text{N}}{0.0250\ \text{kg/m}}} = 88.5\ \text{m/s}$$

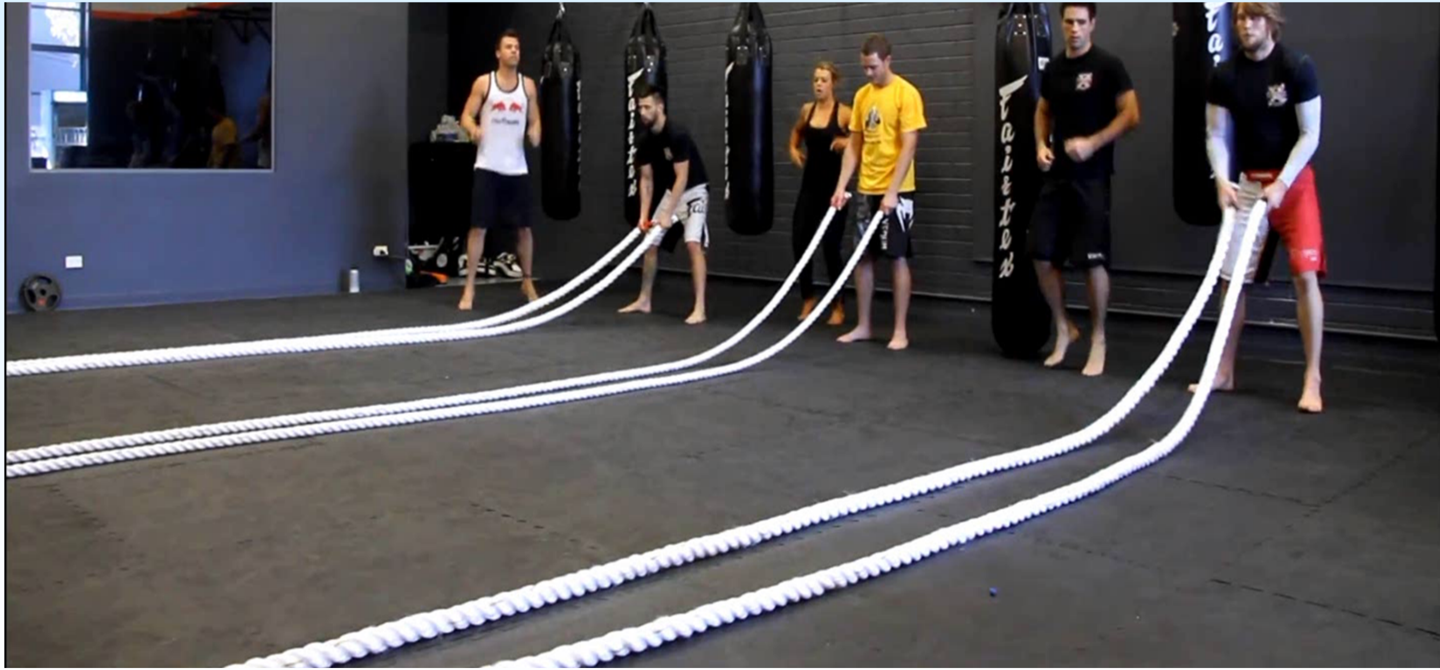
the wavelength is

$$\lambda = \frac{v}{f} = \frac{88.5\ \text{m/s}}{2.00\ \text{s}^{-1}} = 44.3\ \text{m}$$

There are $(80.0\ \text{m})/(44.3\ \text{m}) = 1.81$ wavelengths (that is, cycles of the wave) in the rope.

Part 9. Power

How much work is done every second ?



Mechanical waves: Power

Wave power (P): The instantaneous rate at which energy is transfered along the wave. (P = energy per unit time)

Units: W or J/s

The power in general : $P = \vec{F} \cdot \vec{v}$ (instantaneous rate at which force $\vec{F}$ does work on a particle)

Power along the wave (P): $P(x, t) = F_y(x, t) v_y(x, t)$

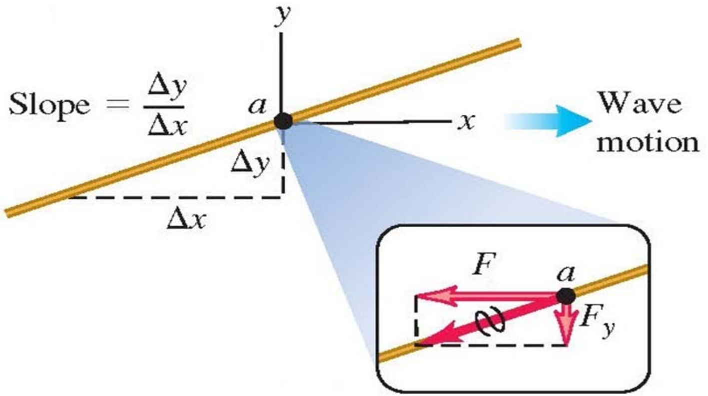
because y is the only direction where the speed of the string is not zero

Mechanical waves: Power

$$P = \vec{F} \cdot \vec{v}$$

$$P(x,t) = F_y(x,t) v_y(x,t)$$

$$F_y(x,t) = -F \frac{\partial y(x,t)}{\partial x}$$



$$P(x,t) = F_y(x,t) v_y(x,t) = -F \frac{\partial y(x,t)}{\partial x} \frac{\partial y(x,t)}{\partial t}$$

$$y(x,t) = A\cos(kx - \omega t)$$

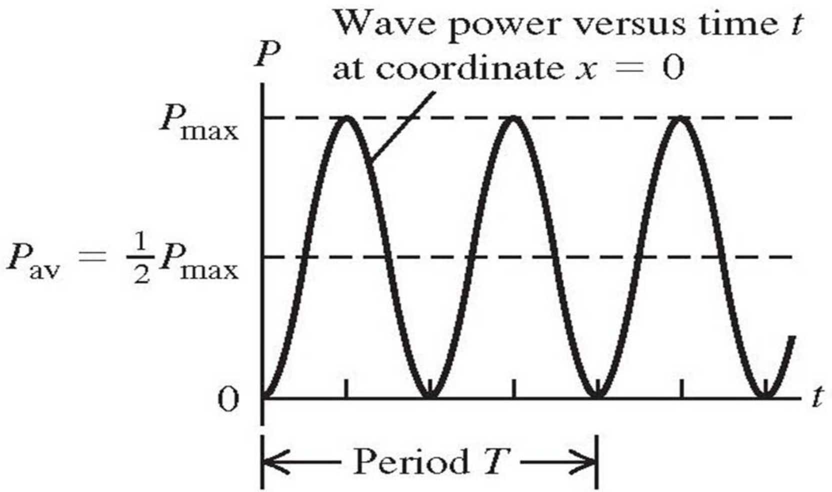
$$\frac{\partial y(x,t)}{\partial x} = -kA\sin(kx - \omega t)$$

$$\frac{\partial y(x,t)}{\partial t} = \omega A\sin(kx - \omega t)$$

The power of the wave:

$$P(x,t) = Fk\omega A^2 \sin^2(kx - \omega t)$$

Mechanical waves: Power



The wave power:

$$P(x, t) = Fk\omega A^2 \sin^2(kx - \omega t)$$

$$P_{max} = Fk\omega A^2 = \sqrt{\mu F}\,\omega^2 A^2$$

&

$$P_{av} = \frac{1}{2} Fk\omega A^2 = \frac{1}{2}\sqrt{\mu F}\,\omega^2 A^2$$

$$v = \sqrt{\frac{F}{\mu}}$$

$$v = \frac{\omega}{k}$$

$$k = \frac{\omega}{\sqrt{\frac{F}{\mu}}}$$

Part 10. Problems

$$\frac{1}{n}\sin x = ?$$

$$\frac{1}{n}\sin x =$$

$$six = 6$$

Mechanical waves: Problems

A: Amplitude = 0.075 m

f: Frequency = 1 / T = 2.00 Hz

v: Wave speed = λ / T = 12.0 m/s

T: Period = 1 / f = 0.5 s

λ: Wavelength = v T = 6.00 m

ω: Angular frequency = 2 π f = 4π

k: Wave number = 2 π / λ = ⅓π

μ: Linear mass density = 0.250 kg/m

F: Tension = 36.0 N

You swing a rope up and down and create a sine wave with the frequency 2.00 Hz, the amplitude 0.075 m and the wave speed 12.0 m/s. The rope weighs 250 grams per meter and is tensioned with the force of 36.0 N.

Calculate the maximum power and average power needed.

$$P_{max} = \sqrt{\mu F}\omega^2 A^2$$

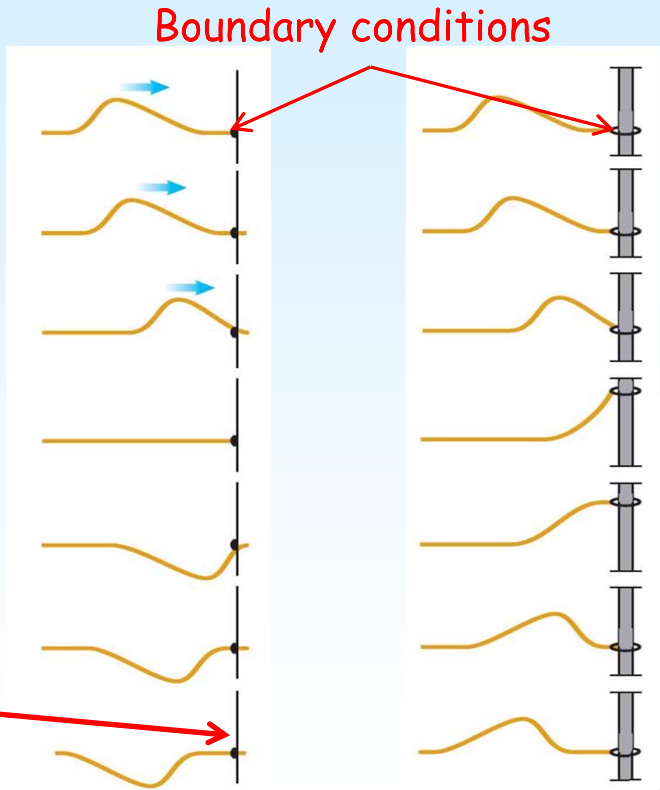
$$= \sqrt{(0.250\ \mathrm{kg/m})(36.0\ \mathrm{N})}(4.00\pi\ \mathrm{rad/s})^2(0.075\ \mathrm{m})^2$$

$$= 2.66\ \mathrm{W}$$

$$P_{av} = \frac{1}{2}P_{max} = \frac{1}{2}(2.66\ \mathrm{W}) = 1.33\ \mathrm{W}$$

Part 11. Reflection of waves

Reflections of a wave



The support provides an opposite force which produces and inverted wave.

Mechanical waves: Reflections

The wavefunction of two waves is typically the sum of the individual wavefunctions.

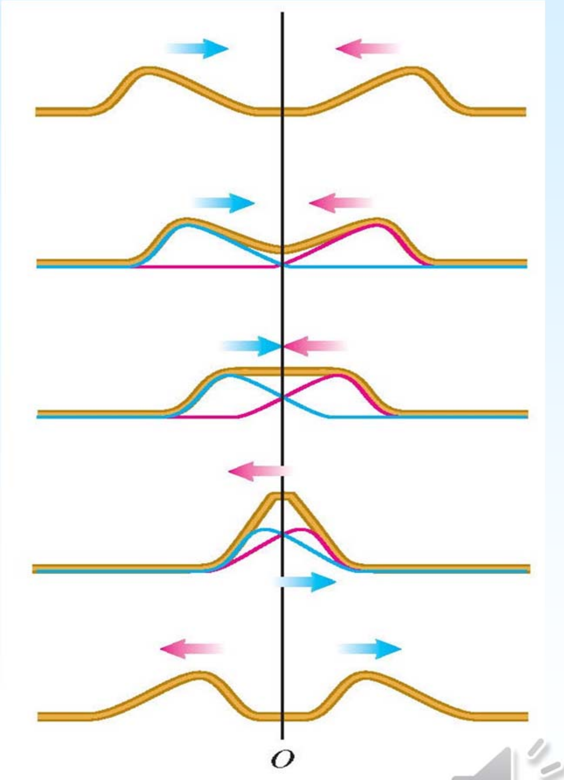
$$y(x, t) = y_1(x, t) + y_2(x, t)$$

This is called the principle of superposition.

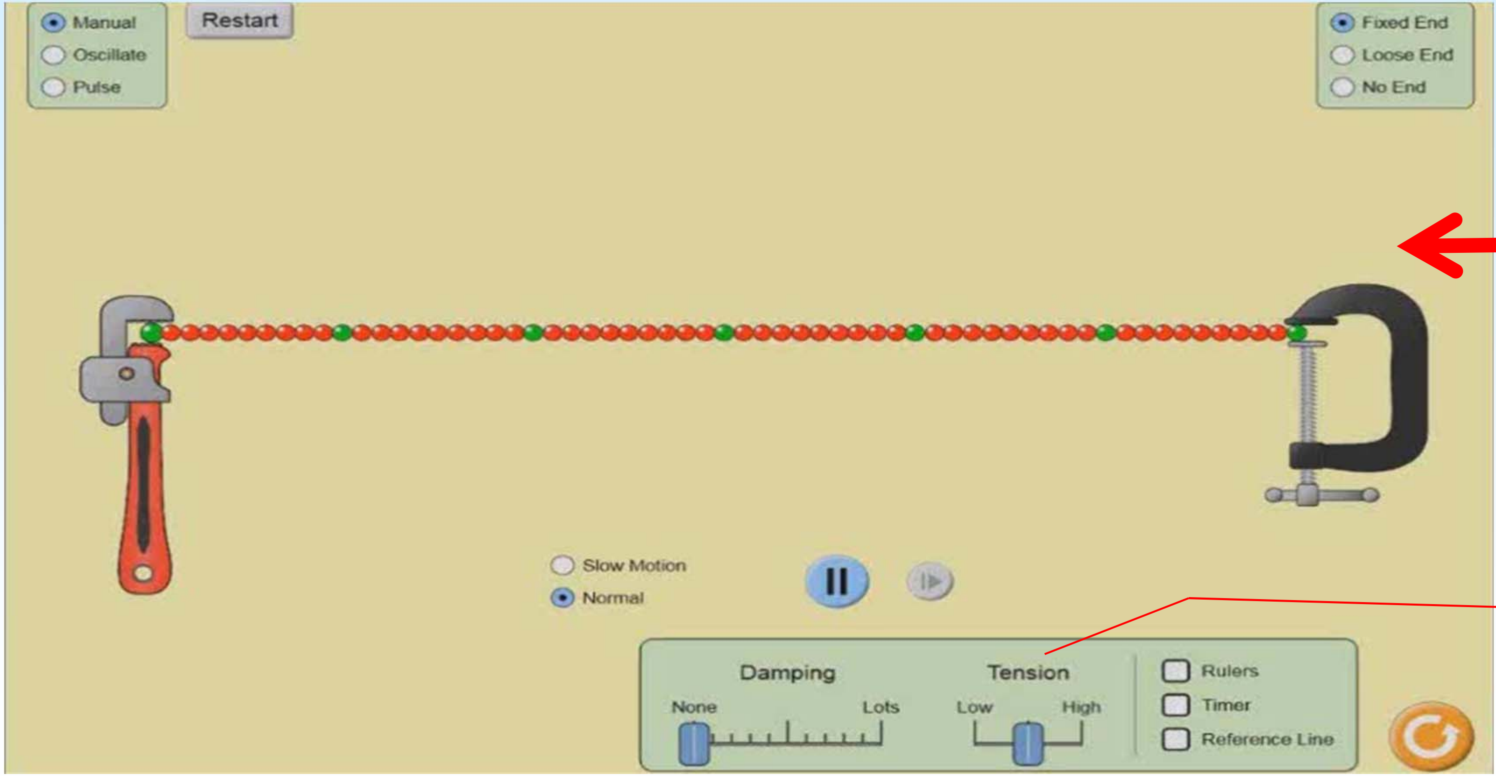
This is true if the wave equations for the waves are linear (they contain the function y(x,t) only to the first power).

For example can sinusoidal waves be superimposed like this because their wave equation is linear.

$$\frac{\partial^2 y(x, t)}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y(x, t)}{\partial t^2}$$



Mechanical waves: Reflections



Different boundary conditions

$$v = \sqrt{\frac{F}{\mu}}$$

https://phet.colorado.edu/en/simulation/wave-on-a-string

Part 12. Standing waves

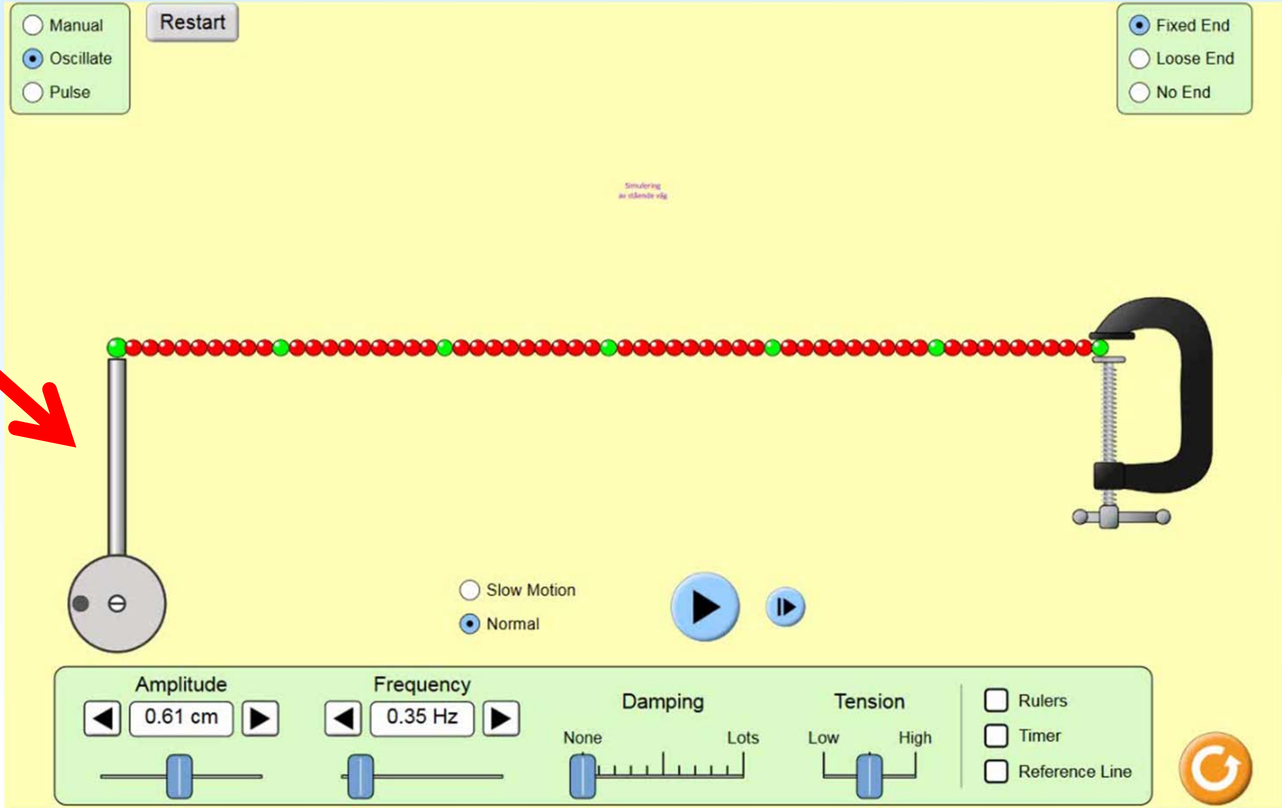
https://www.youtube.com/watch?v=NpEevfOU4Z8

Mechanical waves: Standing waves

https://www.youtube.com/watch?v=-gr7KmTOrx0

Mechanical waves: Standing waves

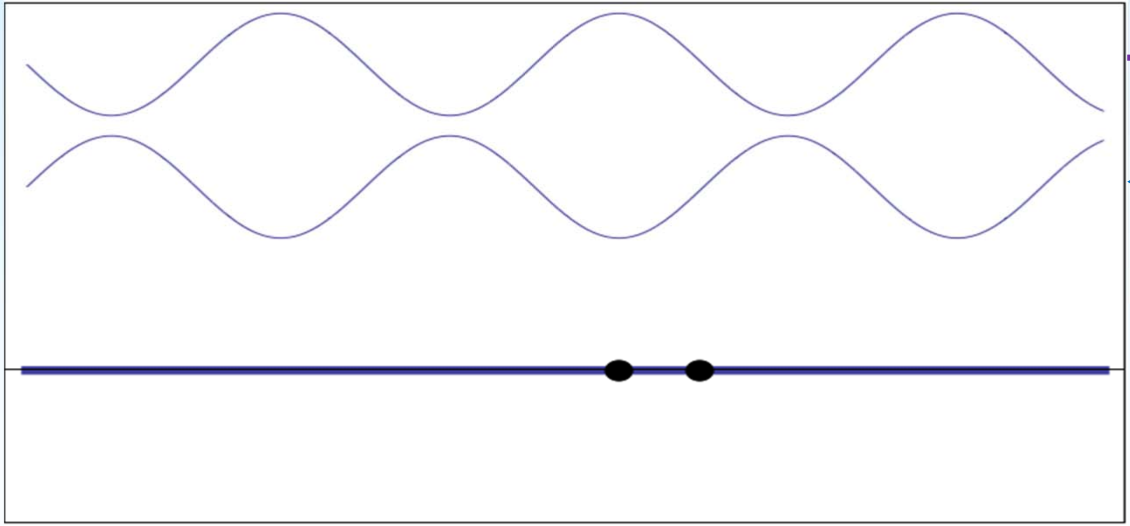
Harmonic oscillation



Mechanical waves: Standing waves

Two waves with the same frequency and wavelength pass each other:

http://www.acs.psu.edu/drussell/Demos/superposition/superposition.html



$$y_2(x,t) = A\cos(kx - \omega t)$$

$$y_1(x,t) = -A\cos(kx + \omega t)$$

$$y(x,t) = y_1(x,t) + y_2(x,t) = A[-\cos(kx + \omega t) + \cos(kx - \omega t)]$$

Mechanical waves: Standing waves

Superposition of two waves:

$$y(x,t) = y_1(x,t) + y_2(x,t) = A[-\cos(kx+\omega t) + \cos(kx-\omega t)]$$

+

Trigonometry: $\cos(a \mp b) = \cos a \cos b \pm \sin a \sin b$

=

$$Y(x,t)=A[-\cos(kx)\cos(\omega t)+\sin(kx)\sin(\omega t) +\cos(kx)\cos(\omega t)+\sin(kx)\sin(\omega t)]$$

=

The wavefunction for a standing wave:

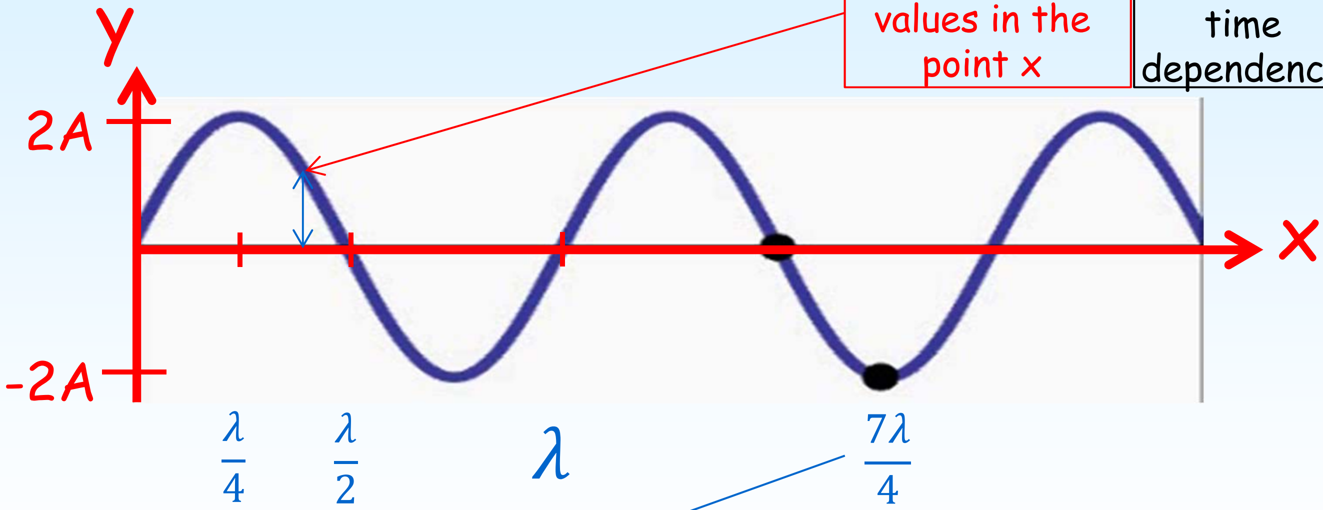
$$y(x,t) = y_1(x,t) + y_2(x,t) = 2A \sin kx \ \sin \omega t$$

Mechanical waves: Standing waves

$$y(x,t) = 2A\sin(kx)\sin(\omega t) = 2A\sin\left(\frac{2\pi}{\lambda}x\right)\sin(\omega t)$$

Gives max-min values in the point x

Gives the time dependence



$$y\left(\frac{7\lambda}{4},t\right) = 2A\sin\left(\frac{2\pi}{\lambda}\frac{7\lambda}{4}\right)\sin(\omega t) = 2A\sin\left(\frac{7\pi}{2}\right)\sin(\omega t) = -2A\sin(\omega t)$$

Nodes:

$$y(x,t) = y_1(x,t) + y_2(x,t) = 2A\sin kx \, \sin\omega t$$

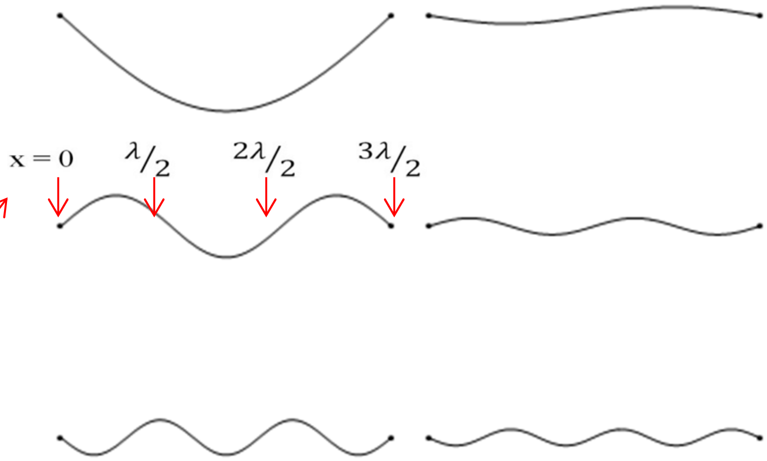
The nodes are given by **sin(kx) = 0**

kx = 0, π, 2π, 3π, 4π,

$$x = 0, \frac{\pi}{k}, \frac{2\pi}{k}, \frac{3\pi}{k}, \frac{4\pi}{k},$$

$$x = 0, \frac{\lambda}{2}, \frac{2\lambda}{2}, \frac{3\lambda}{2}, \frac{4\lambda}{2}, \quad \text{since } \boldsymbol{k = \frac{2\pi}{\lambda}}$$

$$x = 0, \frac{v}{2f}, \frac{2v}{2f}, \frac{3v}{2f}, \frac{4v}{2f}, \quad \text{since } \boldsymbol{\lambda = \frac{v}{f}}$$



What is the velocity and accelerationen ?

Displacement:

$$y(x,t) = 2A \sin(kx) \sin(\omega t)$$

Wavefunction

Velocity:

$$v_y(x,t) = \frac{\partial y(x,t)}{\partial t} \rightarrow v_y(x,t) = 2A\omega \sin(kx) \cos(\omega t)$$

Acceleration:

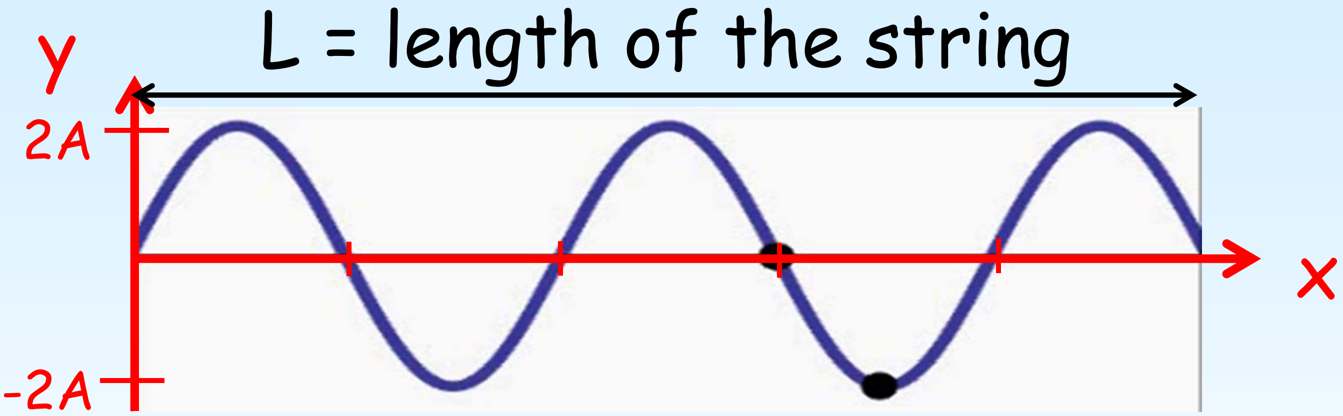
$$a_y(x,t) = \frac{\partial v_y(x,t)}{\partial t} = \frac{\partial^2 y(x,t)}{\partial t^2} \rightarrow a_y(x,t) = -2A\omega^2 \sin(kx) \sin(\omega t)$$

Part 13. String instruments

Octobasse violin

https://www.youtube.com/watch?v=12X-i9YHzmE

Example of a standing wave on a string:



Nodes: $0 \quad \frac{\lambda}{2} \quad \lambda \quad \frac{3\lambda}{2} \quad 2\lambda \quad \frac{5\lambda}{2} = L$

$$\lambda = \frac{2L}{5}$$

$$\lambda = \frac{v}{f}$$

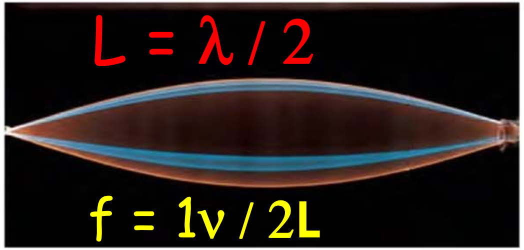
$$f = \frac{5v}{2L}$$

The speed of the waves that build up the standing wave.

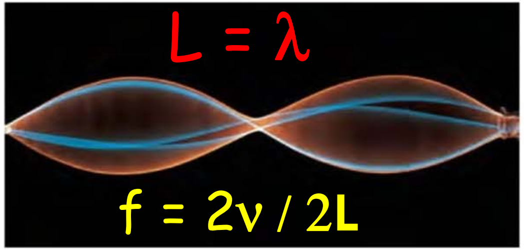
Mechanical waves: String instruments

L = length of the string

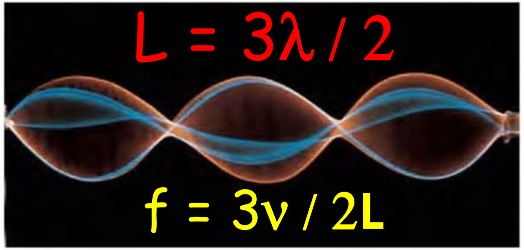
(a) String is one-half wavelength long.



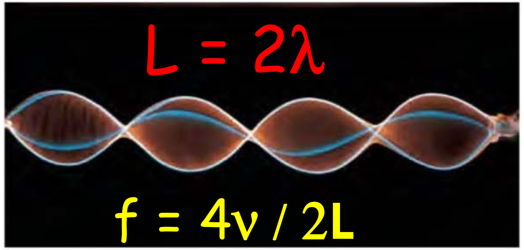
(b) String is one wavelength long.



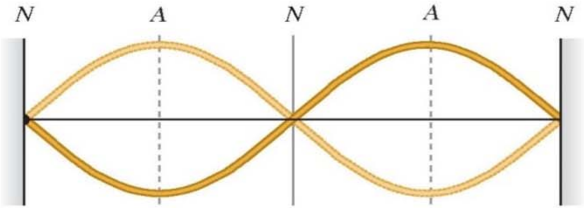
(c) String is one and a half wavelengths long.



(d) String is two wavelengths long.



$$f = v / \lambda$$



N = **nodes:** points at which the string never moves

A = **antinodes:** points at which the amplitude of string motion is greatest

Node Antinode Node Antinode Node

Mechanical waves: String instruments

Strings of length L having nodes at both ends:

Nodes when $\sin(kx) = 0$

$$x = 0, \frac{\pi}{k}, \frac{2\pi}{k}, \frac{3\pi}{k}, \ldots$$

$$= 0, \frac{\lambda}{2}, \frac{2\lambda}{2}, \frac{3\lambda}{2}, \ldots$$

$$L = n\frac{\lambda}{2} \quad (n = 1, 2, 3, \ldots)$$

$$\lambda_n = \frac{2L}{n} \quad (n = 1, 2, 3, \ldots)$$

$$\lambda = 2L / n = v / f$$

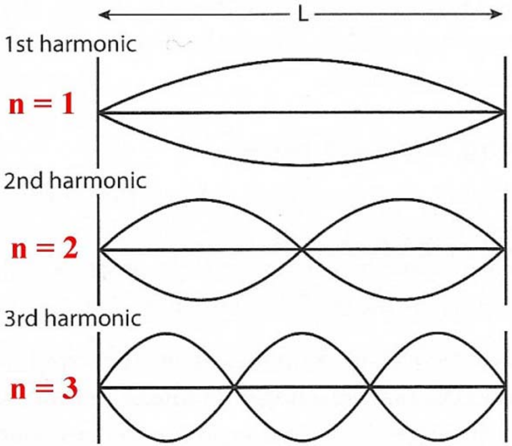
$$f_n = n\frac{v}{2L} = nf_1 \quad (n = 1, 2, 3, \ldots)$$

f_1, f_2, f_3.... Harmonic frequencies

f_1: Fundamental frequency

f_2, f_3, f_4.... Overtones

$\lambda_n = \frac{2}{n}L$ $\quad f_n = \frac{v}{\lambda_n}$ ← samma för alla n



En halv period

$\lambda_1 = \frac{2}{1}L \quad f_1 = \frac{v}{\lambda_1} = \frac{v}{2L}$

En hel period

$\lambda_2 = \frac{2}{2}L \quad f_2 = \frac{v}{\lambda_2} = \frac{v}{L} = 2f_1$

Tre halva perioder

$\lambda_3 = \frac{2}{3}L \quad f_3 = \frac{v}{\lambda_3} = \frac{v}{2/3L} = 3f_1$

Mechanical waves: String instruments

$$f_1 = v/2L$$

$$v = \sqrt{F/\mu}$$

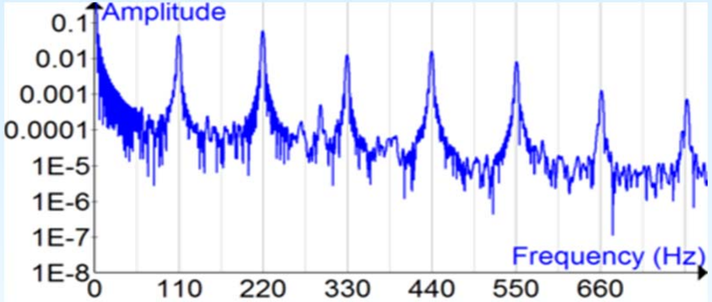
$$f_1 = \frac{1}{2L}\sqrt{\frac{F}{\mu}}$$

Long string:	Low frequency
Thick string:	Low frequency
Large tension:	High frequency

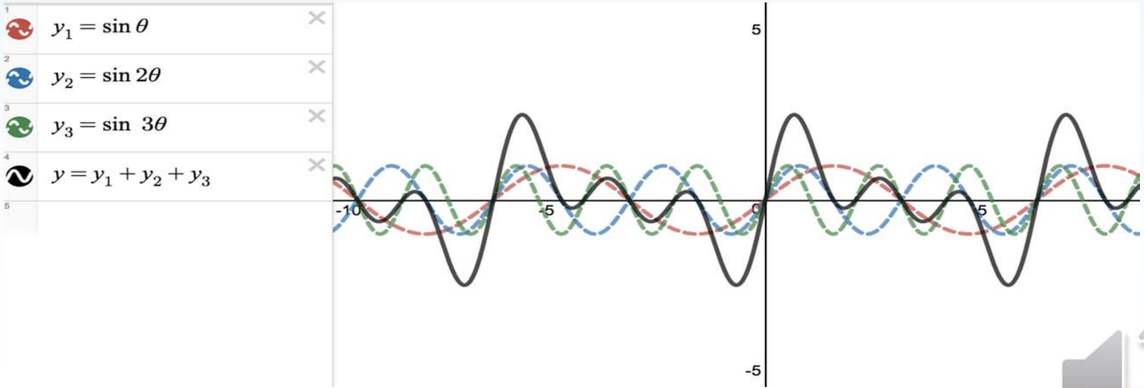
Mechanical waves: String instruments

A string in a stringed instrument normally produces not only a fundamental frequency but an overlay of all harmonic frequencies.

The amplitude of the different frequencies varies:



The resulting wave has a complicated shape:



Mechanical waves: String instruments

Part 14. Problems

$$\frac{1}{n}\sin x = ?$$

$$\frac{1}{\cancel{n}}\,\text{si}\cancel{\text{n}}\,x =$$

$$six = 6$$

Mechanical waves: Problems

A sine wave moves in negative x-direction along a guitar string at the speed of 143 m/s.

The amplitude is 0.750 mm and the frequency 440 Hz.

The wave is reflected at x = 0 and forms a standing wave.

What will be the function that describes the movement of the string in the y-direction ?

$$y(x,t) = 2A\sin(kx)\sin(\omega t)$$

$$A = 0.750\text{ mm} = 7.50 \times 10^{-4}\text{ m}$$

$$\boxed{\omega = 2\pi f} = (2\pi\text{ rad})(440\text{ s}^{-1}) = 2760\text{ rad/s}$$

$$\boxed{k = \frac{\omega}{v}} = \frac{2760\text{ rad/s}}{143\text{ m/s}} = 19.3\text{ rad/m}$$

Mechanical waves: Problems

v = 143 m/s
f = 440 Hz
A = 0.00075 m
ω = 2760 rad/s
k = 19.3 rad/m

Where will there be nodes on the string ?

There will be nodes for $\mathbf{x} = 0, \frac{\lambda}{2}, \frac{2\lambda}{2}, \frac{3\lambda}{2}, \ldots$

$f = v / \lambda \Rightarrow \lambda = v / f = 143 / 440 = 0.325$ m

There will be nodes for x = 0, 0.163 m, 0.325 m,

Mechanical waves: Problems

$v = 143$ m/s
$f = 440$ Hz
$A = 0.00075$ m
$\omega = 2760$ rad/s
$k = 19.3$ rad/m

What is the amplitude of the standing wave ?
What will be the maximum speed and the maximum acceleration ?

$$y(x,t) = 2A \sin(kx) \sin(\omega t)$$

→ Amplitude = 2A = 0.0015 m

$$v_y(x,t) = 2A\omega \sin(kx) \cos(\omega t)$$

$$v_y(x,t)_{max} = 2A\omega = 4.14 \text{ m/s}$$

$$a_y(x,t) = -2A\omega^2 \sin(kx) \sin(\omega t)$$

$$a_y(x,t)_{max} = 2A\omega^2 = 11426 \text{ m/s}^2$$

Mechanical waves: Problems

An octobasse has a string that is 2.50 m long and weighs 40.0 grams per meter.

What tension force is needed for the fundamental frequency to be 20.0 Hz ?

$$v = \sqrt{\frac{F}{\mu}} \qquad f_n = \frac{nv}{2L}$$

$$f_1 = \frac{1}{2L}\sqrt{\frac{F}{\mu}}$$

$$F = 4\mu L^2 f_1^{\,2} = 4(40.0 \times 10^{-3}\,\mathrm{kg/m})(2.50\,\mathrm{m})^2(20.0\,\mathrm{s}^{-1})^2 = 400\ \mathrm{N}$$

Mechanical waves: Problems

$f_1 = 20.0$ Hz
$L = 2.50$ m
$\mu = 40.0$ g/m
$F = 400$ N

What will be the frequency and wavelength of the second harmonic frequency ?

What will be the frequency and wavelength of the second overtone ?

$$f_n = n\frac{v}{2L} = nf_1 \quad (n = 1, 2, 3, \ldots)$$

$$\lambda_n = \frac{2L}{n} \quad (n = 1, 2, 3, \ldots)$$

$$f_2 = 2f_1 = 2(20.0\ \text{Hz}) = 40.0\ \text{Hz}$$

$$\lambda_2 = \frac{2L}{2} = \frac{2(2.50\ \text{m})}{2} = 2.50\ \text{m}$$

The second overtone is the second frequency above the fundamental frequency, i.e. n = 3

$$f_3 = 3f_1 = 3(20.0\ \text{Hz}) = 60.0\ \text{Hz}$$

$$\lambda_3 = \frac{2L}{3} = \frac{2(2.50\ \text{m})}{3} = 1.67\ \text{m}$$

Mechanical waves: Problems

$f_1 = 20.0$ Hz
$L = 2.50$ m
$\mu = 40.0$ g/m
$F = 400$ N
$\lambda_1 = 1.25$ m

The string vibrates at its fundamental frequency.

What is the frequency and wavelength of the sound it emits ?

The speed of sound is 344 m/s.

$$v = \lambda / T = \lambda f$$
$$\lambda = v / f$$

$$f = f_1 = 20.0 \text{ Hz}$$
$$\lambda_{1(\text{sound})} = \frac{v_{\text{sound}}}{f_1} = \frac{344 \text{ m/s}}{20.0 \text{ Hz}} = 17.2 \text{ m}$$

Part 15. Summary

Mechanical waves: Summary

The sinusoidal oscillations on a string are described by the wave equation

$$\frac{\partial^2 y(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y(x,t)}{\partial t^2}$$

which has the wavefunction as a solution

$$y(x,t) = A\cos(kx-\omega t)$$

Velocity and acceleration are obtained by derivation

$$v_y(x,t) = \frac{\partial y(x,t)}{\partial t} = \omega A \sin(kx - \omega t)$$

$$a_y(x,t) = \frac{\partial^2 y(x,t)}{\partial t^2} = -\omega^2 A \cos(kx - \omega t) = -\omega^2 y(x,t)$$

Wave speed

$$v = \lambda/T = \omega/k \qquad v = \sqrt{\frac{F}{\mu}}$$

Mechanical waves: Summary

Average power → $$P_{av} = \frac{1}{2}\mu(\omega A)^2 v = \frac{1}{2}\sqrt{\mu F}(\omega A)^2$$

The power function → $$P(x,t) = 2P_{av}\sin^2(kx-\omega t)$$

Wavefunction for a standing wave → $$y(x,t) = 2A\sin(kx)\sin(\omega t)$$

Fundamental frequency → $$f_1 = \frac{1}{2L}\sqrt{\frac{F}{\mu}}$$ $f_n = nf_1 \quad n = 2, 3, 4\ldots$