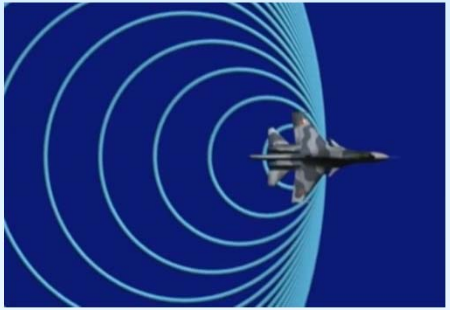
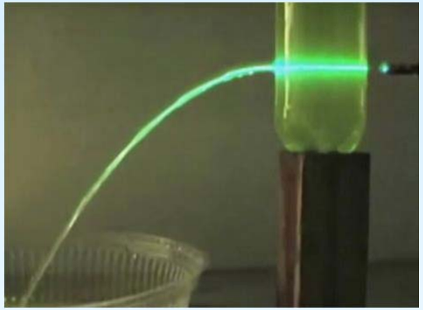
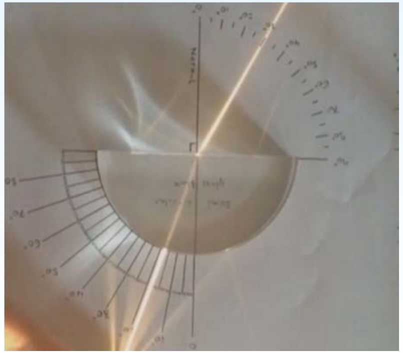
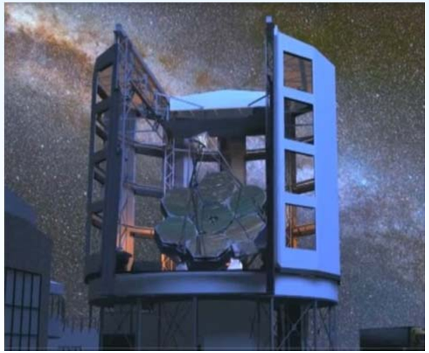
Wavemechanics and optics





Chapter 15 – Mechanical waves

Vincent Hedberg - Lunds Universitet

Transverse waves

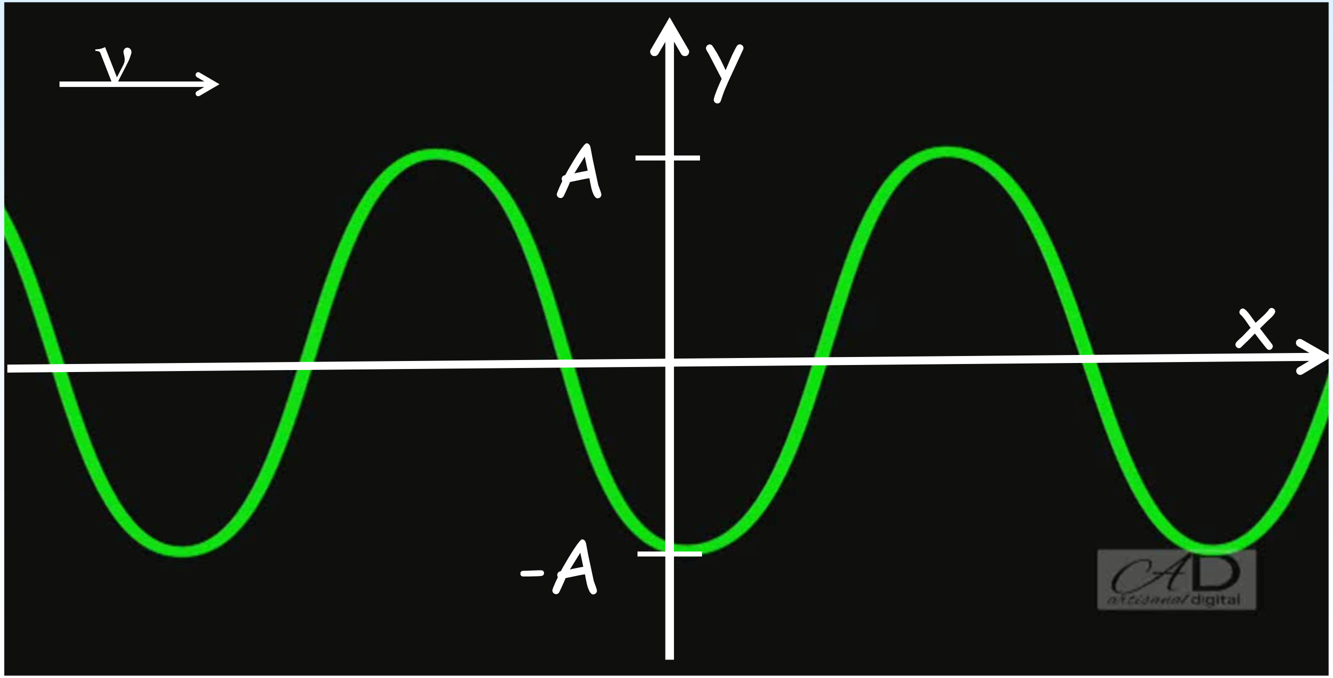
Mechanical waves: Transverse waves

Transverse wave: The medium moves transverse to the wave direction.

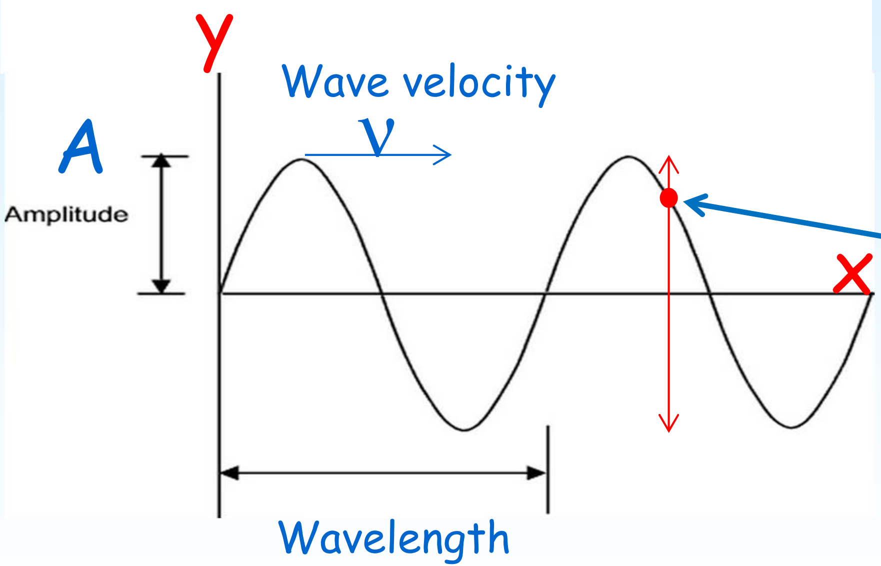
https://www.youtube.com/watch?v=FUBGrH-PbsU

Mechanical waves: Transverse waves

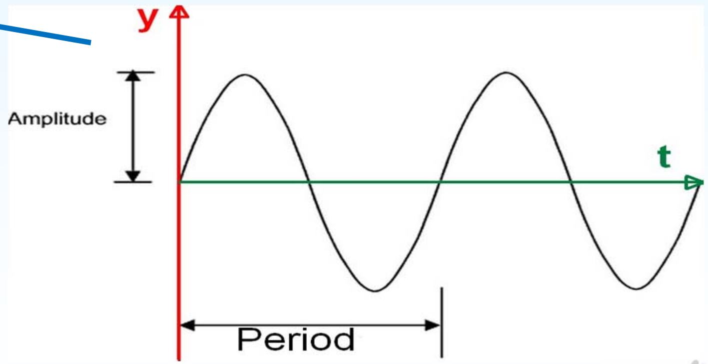
A special transverse wave is the sinusoidal wave:



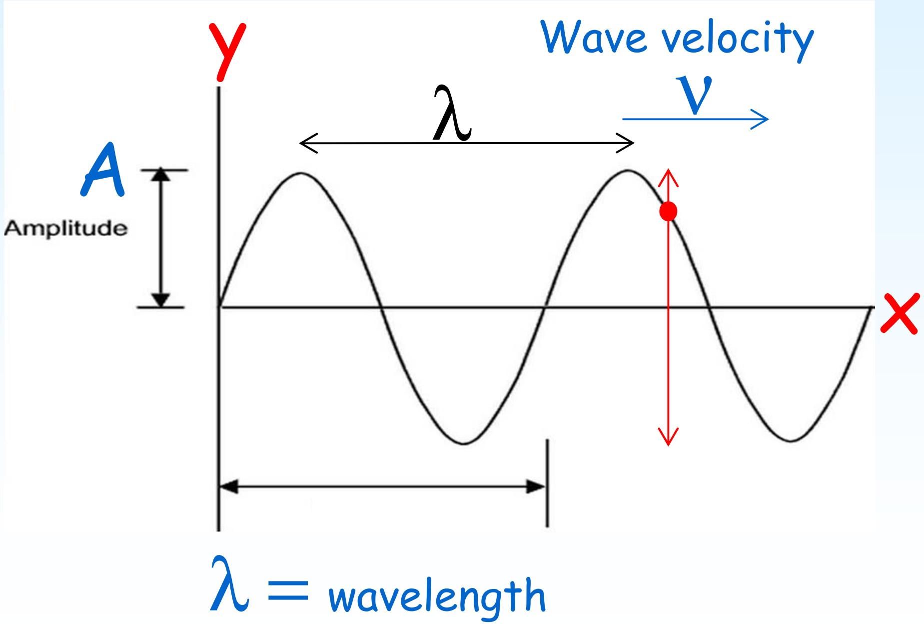
Transverse sinusoidal waves



Every point on the wave moves up and down like an harmonic oscillator with the period T.



Definitions:



A: Amplitude (m)

T: Period (s)

λ: Wavelength (m)

v: Wave speed (m/s) = λ / T

f: Frequency (Hz) = $1 / T$

ω: Angular frequency (radians/s) = $2 \pi f$

k: Wave number (radians/m) = $2 \pi / \lambda$

Longitudinal waves

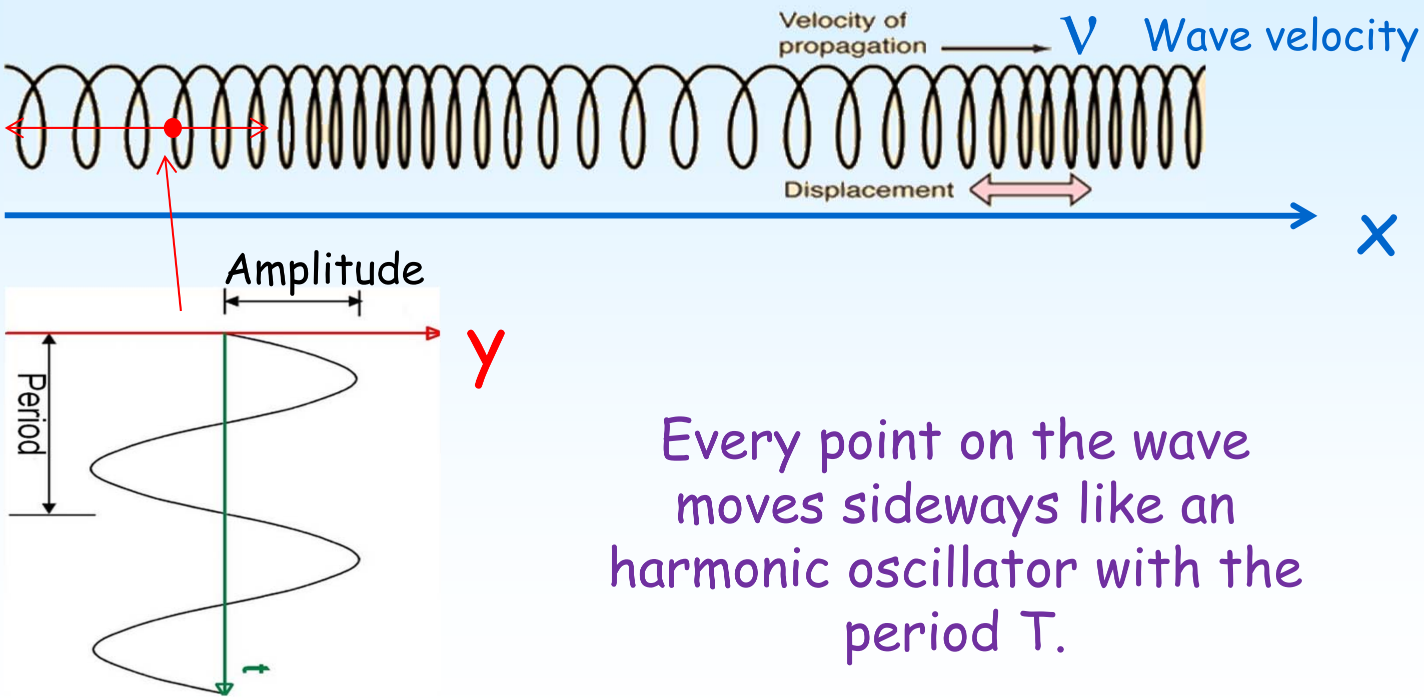
Japanese earthquake

Simulation of Japanese earthquake

Longitudinal waves:

The medium moves in the wave direction.

Longitudinal sinusoidal waves

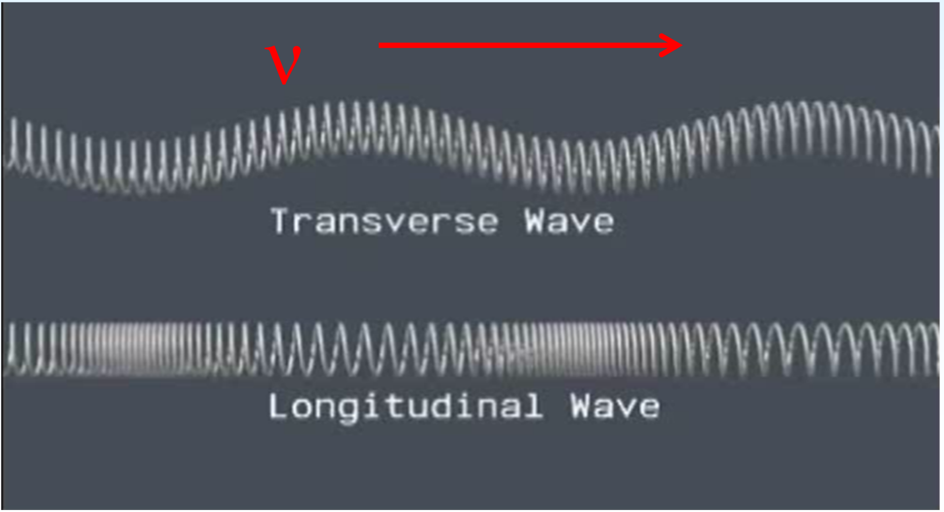


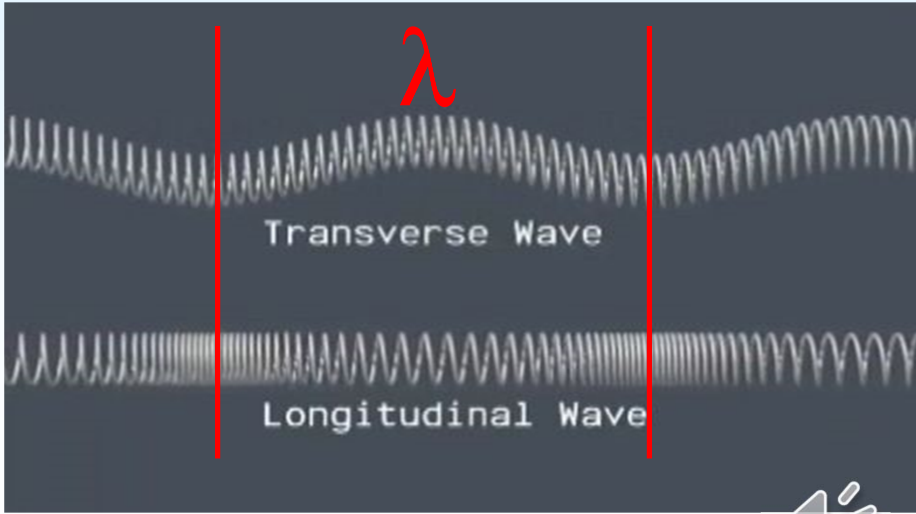
Every point on the wave moves sideways like an harmonic oscillator with the period T.

Mechanical waves: Longitudinal waves

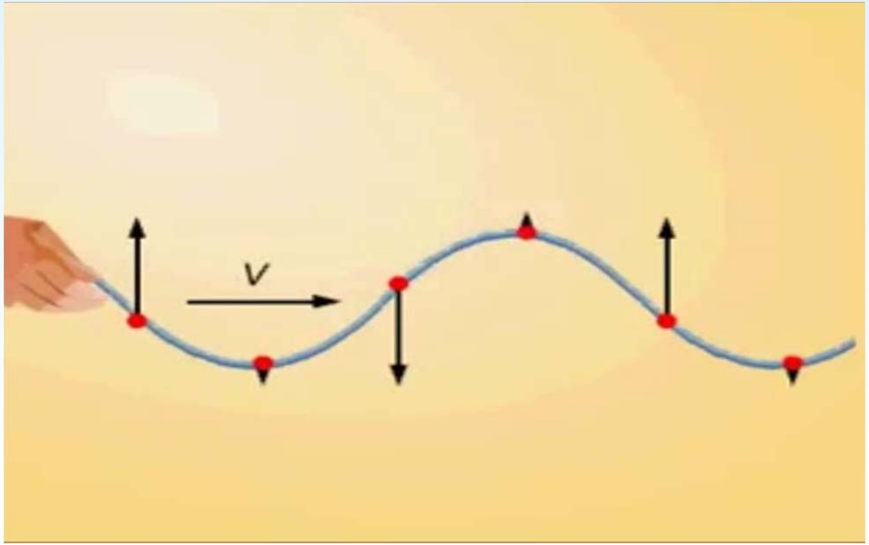
What is the wavelength (λ) for a sinusoidal wave ?
What is the wave velocity (v) ?

$$v = \lambda / T = \lambda f$$





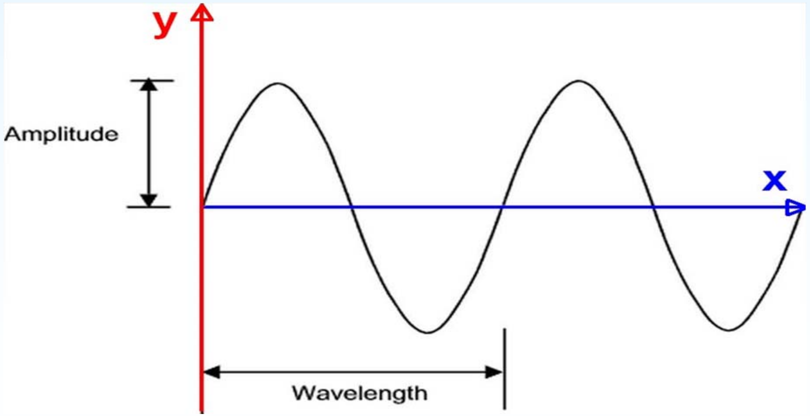
The wavefunction



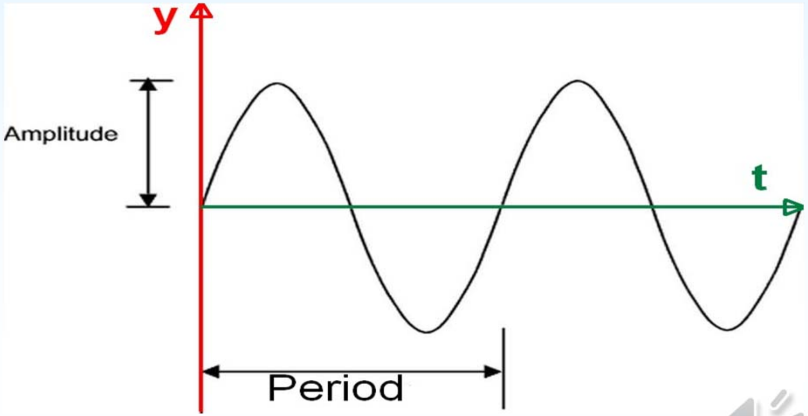
Mechanical waves: The wavefunction

The wavefunction y(x,t): The wave function describes the height of the wave as a function of both distance and time.

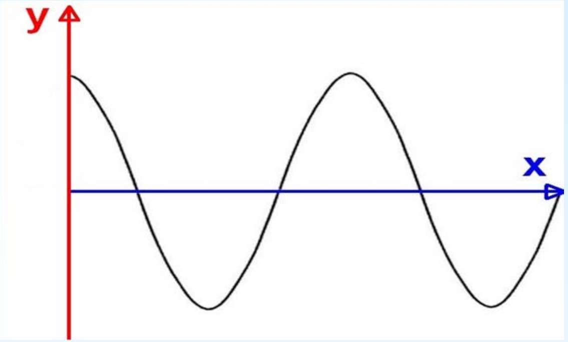
The height of the wave as a function of distance x:



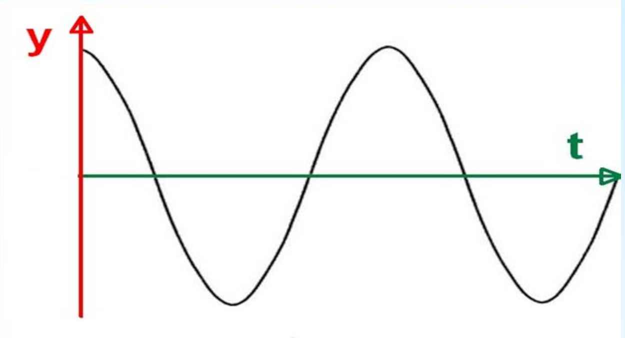
The height of the wave as a function of time t:



Mechanical waves: The wavefunction



$$y(x, t = 0) = A\cos kx$$



$$y(x = 0, t) = A\cos \omega t$$

$$y(x, t) = A\cos(kx - \omega t)$$ (sinusoidal wave moving in $+x$-direction)

+ if moving in the negative x direction

Mechanical waves: The wavefunction

$$y(x, t) = A\cos(kx - \omega t)$$ (sinusoidal wave moving in $+x$-direction)

Amplitude: A

Wavenumber: $k = \frac{2\pi}{\lambda}$

$v = \lambda / T$
$f = 1 / T$

Angular frequency: $\omega = \frac{2\pi}{T}$

$v = \lambda / T = (2\pi/k) / (2\pi/\omega) = \omega / k$

Mechanical waves: The wavefunction

The wavefunction:

$$y(x, t) = A\cos(kx - \omega t)$$

The velocity:

$$v_y(x, t) = \frac{\partial y(x, t)}{\partial t} = \omega A \sin(kx - \omega t)$$

The acceleration:

$$a_y(x, t) = \frac{\partial^2 y(x, t)}{\partial t^2} = -\omega^2 A\cos(kx - \omega t) = -\omega^2 y(x, t)$$

The wave equation:

$$\frac{\partial^2 y(x, t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y(x, t)}{\partial t^2}$$

Wave velocity:

$$v = \lambda / T = \omega / k$$

Wave speed and string properties

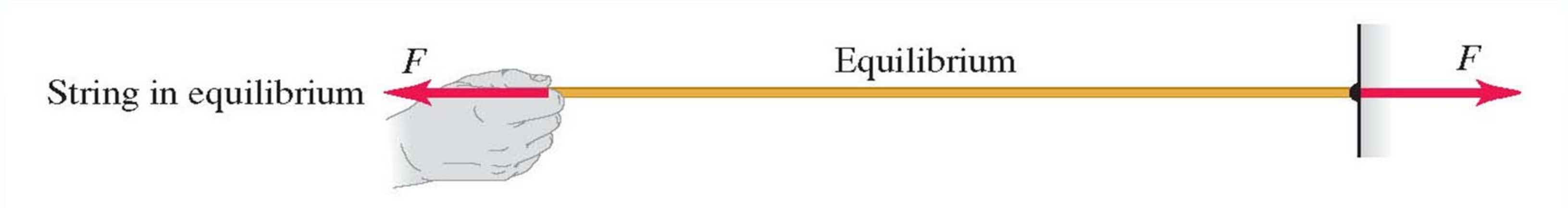
https://www.youtube.com/watch?v=ttgLyWFINJI

Mechanical waves: Wave speed

The wave speed on a string depends on two things:

$$v = \sqrt{\frac{F}{\mu}}$$

- F: The string tension
- μ: The mass of the string per unit length



More generally:

$$v = \sqrt{\frac{\text{Restoring force returning the system to equilibrium}}{\text{Inertia resisting the return to equilibrium}}}$$

Power

How much work is done every second ?

Mechanical waves: Power

Wave power (P): The instantaneous rate at which energy is transfered along the wave. (P = energy per unit time)
Units: W or J/s

The power in general :

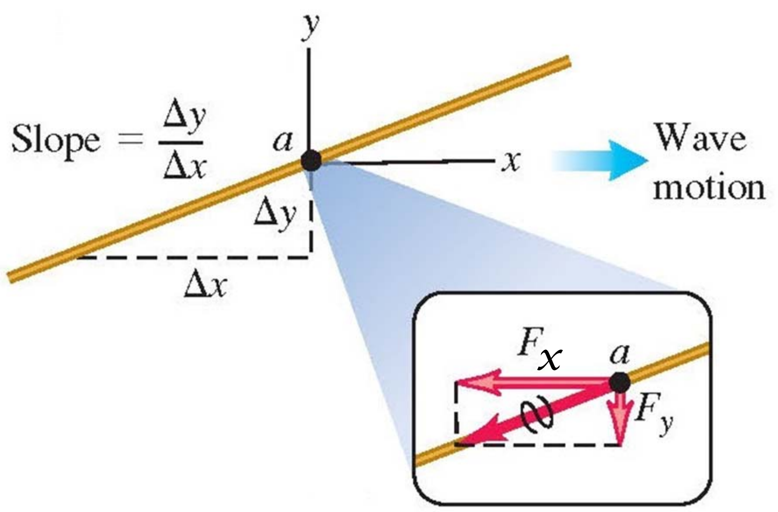
$$P = \vec{F} \cdot \vec{v}$$ (instantaneous rate at which force $\vec{F}$ does work on a particle)

Power along the wave (P):

$$P(x, t) = F_y(x, t)v_y(x, t)$$

because y is the only direction where the speed of the string is not zero

Wave on a string



The ratio of the force in the y-direction to the force in the x-direction is given by the slope of the string which can be calculated by derivation:

$$Slope = \frac{\Delta y}{\Delta x} = \frac{F_y}{F_x} = \frac{dy}{dx}$$

$$F_y(x,t) = -F_x \frac{\partial y(x,t)}{\partial x}$$

F_y is in negative y-direction

F_x = F is the string tension

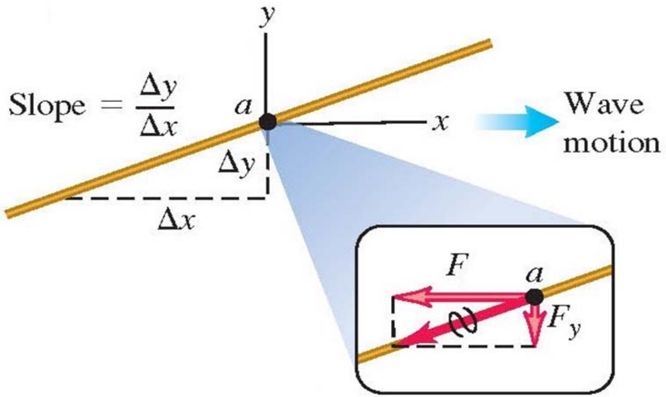
Mechanical waves: Power

$$P = \vec{F} \cdot \vec{v}$$

$$P(x, t) = F_y(x, t) v_y(x, t)$$

$$F_y(x, t) = -F \frac{\partial y(x, t)}{\partial x}$$

$$P(x, t) = F_y(x, t) v_y(x, t) = -F \frac{\partial y(x, t)}{\partial x} \frac{\partial y(x, t)}{\partial t}$$



$$y(x, t) = A \cos(kx - \omega t)$$

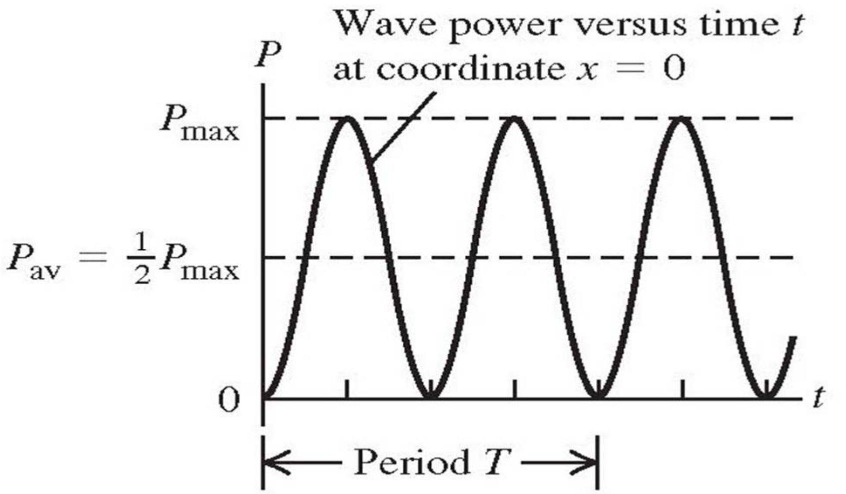
$$\frac{\partial y(x, t)}{\partial x} = -kA \sin(kx - \omega t)$$

$$\frac{\partial y(x, t)}{\partial t} = \omega A \sin(kx - \omega t)$$

The power of the wave:

$$P(x, t) = Fk\omega A^2 \sin^2(kx - \omega t)$$

Mechanical waves: Power



The wave power:

$$P(x,t) = Fk\omega A^2 \sin^2(kx - \omega t)$$

$$P_{max} = Fk\omega A^2 = \sqrt{\mu F}\,\omega^2 A^2$$

&

$$P_{av} = \frac{1}{2}Fk\omega A^2 = \frac{1}{2}\sqrt{\mu F}\,\omega^2 A^2$$

$$v = \sqrt{\frac{F}{\mu}}$$

$$v = \frac{\omega}{k}$$

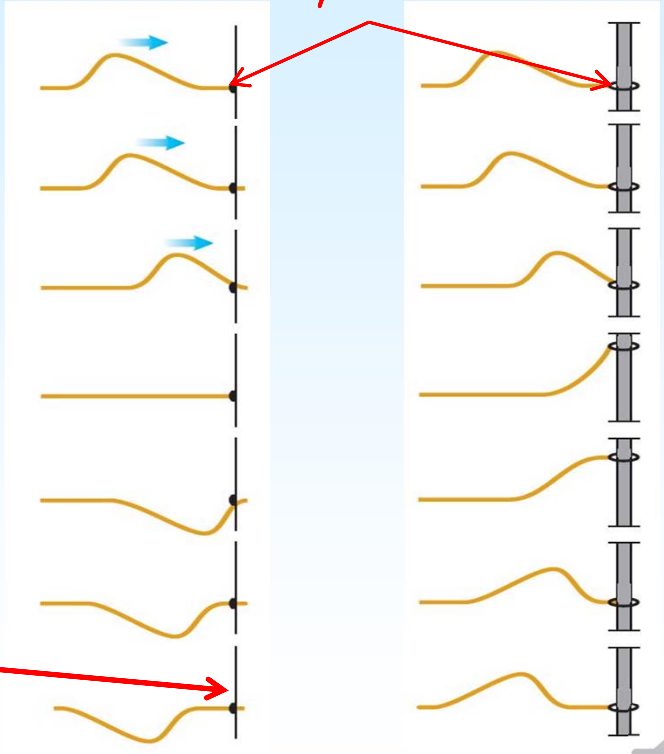
$$k = \frac{\omega}{\sqrt{\frac{F}{\mu}}}$$

Reflection of waves

Mechanical waves: Reflections

Reflections of a wave

Boundary conditions



The support provides an opposite force which produces and inverted wave.

Mechanical waves: Reflections

The wavefunction of two waves is typically the sum of the individual wavefunctions.

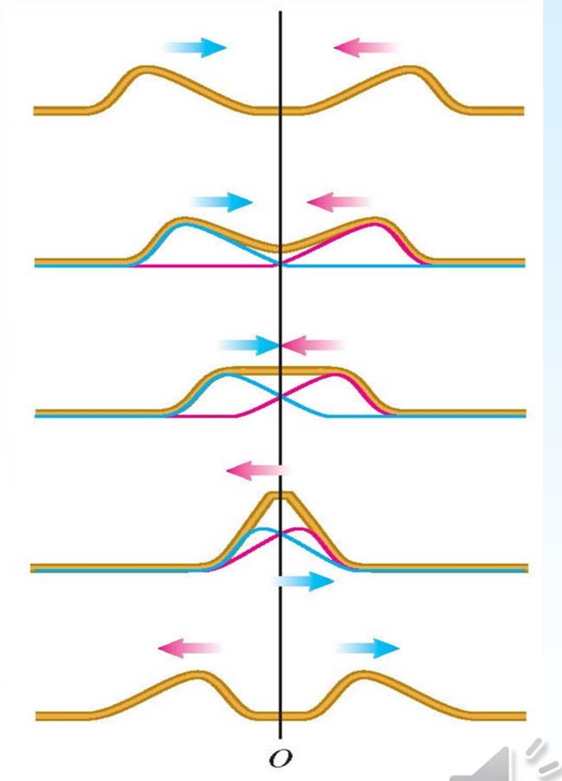
$$y(x, t) = y_1(x, t) + y_2(x, t)$$

This is called the principle of superposition.

This is true if the wave equations for the waves are linear (they contain the function y(x,t) only to the first power).

For example can sinusoidal waves be superimposed like this because their wave equation is linear.

$$\frac{\partial^2 y(x, t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y(x, t)}{\partial t^2}$$



Standing waves

https://www.youtube.com/watch?v=NpEevfOU4Z8

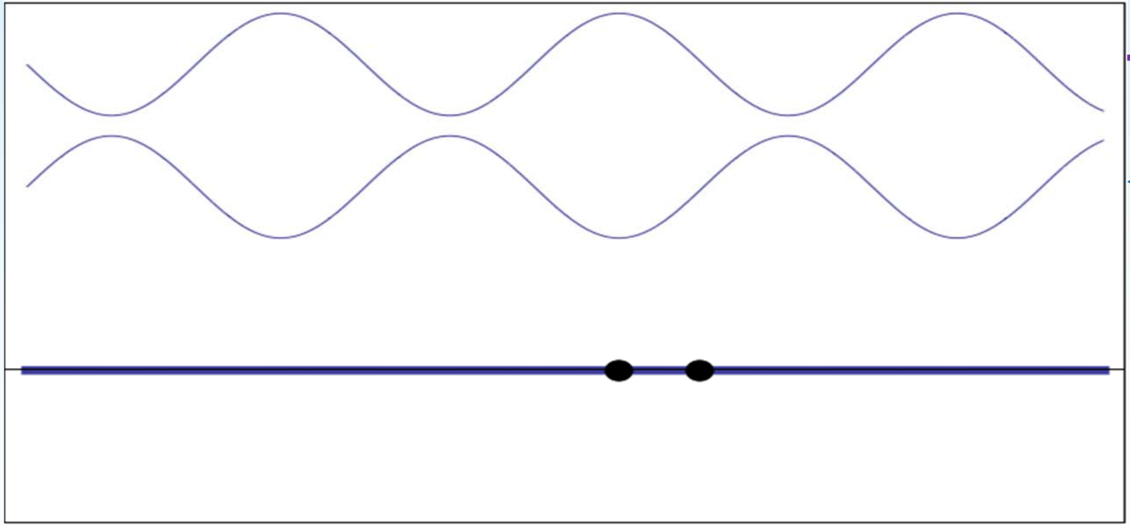
Mechanical waves: Standing waves

https://www.youtube.com/watch?v=-gr7KmTOrx0

Mechanical waves: Standing waves

Two waves with the same frequency and wavelength pass each other:

http://www.acs.psu.edu/drussell/Demos/superposition/superposition.html



$$y_2(x,t) = A\cos(kx - \omega t)$$

$$y_1(x,t) = -A\cos(kx + \omega t)$$

$$y(x,t) = y_1(x,t) + y_2(x,t) = A[-\cos(kx + \omega t) + \cos(kx - \omega t)]$$

Mechanical waves: Standing waves

Superposition of two waves:

$$y(x,t) = y_1(x,t) + y_2(x,t) = A[-\cos(kx+\omega t) + \cos(kx-\omega t)]$$

+

Trigonometry: $\cos(a \mp b) = \cos a \cos b \pm \sin a \sin b$

=

$$Y(x,t)=A[-\cos(kx)\cos(\omega t)+\sin(kx)\sin(\omega t) +\cos(kx)\cos(\omega t)+\sin(kx)\sin(\omega t)]$$

=

The wavefunction for a standing wave:

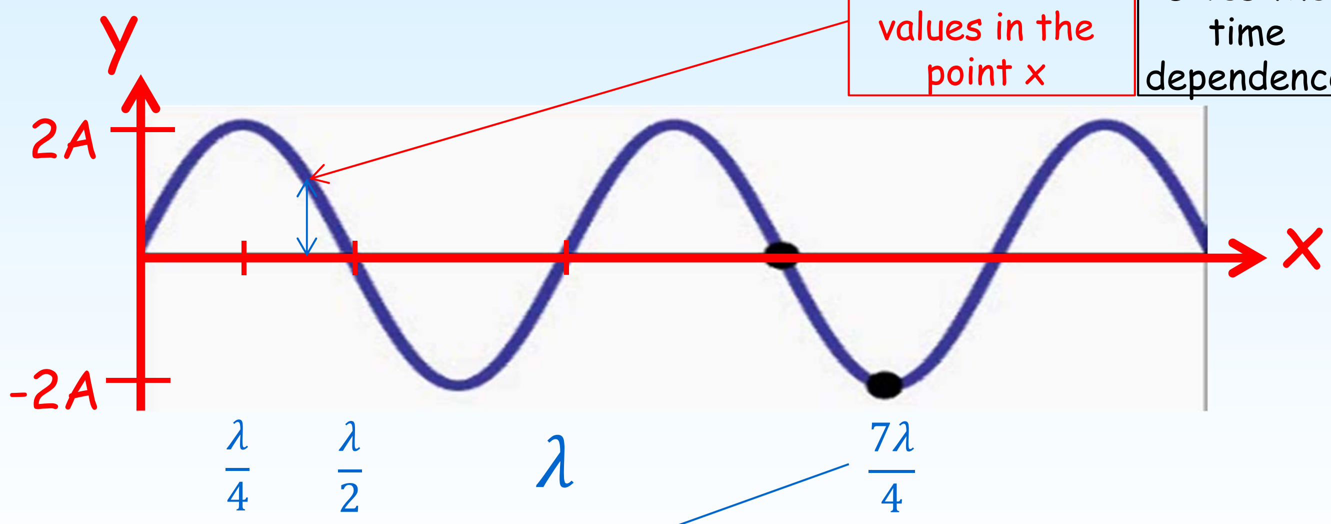
$$y(x,t) = y_1(x,t) + y_2(x,t) = 2A \sin kx \ \sin \omega t$$

Mechanical waves: Standing waves

$$y(x,t) = 2A\sin(kx)\sin(\omega t) = 2A\sin\left(\frac{2\pi}{\lambda}x\right)\sin(\omega t)$$

Gives max-min values in the point x

Gives the time dependence



$$y\left(\frac{7\lambda}{4},t\right) = 2A\sin\left(\frac{2\pi}{\lambda}\frac{7\lambda}{4}\right)\sin(\omega t) = 2A\sin\left(\frac{7\pi}{2}\right)\sin(\omega t) = -2A\sin(\omega t)$$

Nodes:

$$y(x,t) = y_1(x,t) + y_2(x,t) = 2A\sin kx \; \sin\omega t$$

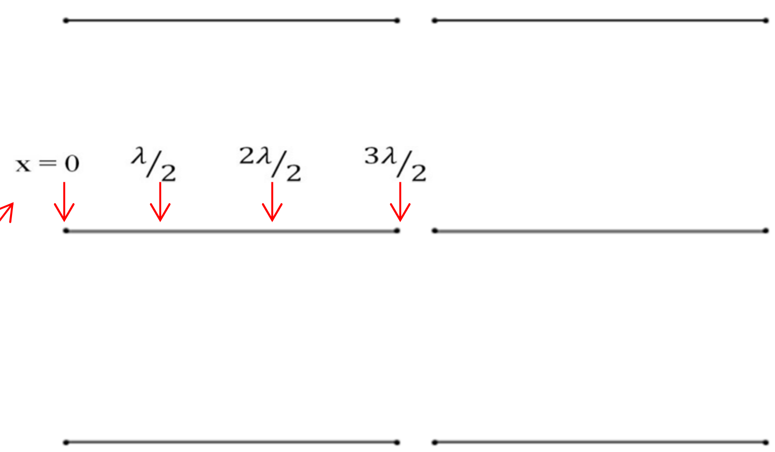
The nodes are given by **sin(kx) = 0**

$kx = 0, \pi, 2\pi, 3\pi, 4\pi,$

$$x = 0, \frac{\pi}{k}, \frac{2\pi}{k}, \frac{3\pi}{k}, \frac{4\pi}{k},$$

$$x = 0, \frac{\lambda}{2}, \frac{2\lambda}{2}, \frac{3\lambda}{2}, \frac{4\lambda}{2}, \quad \text{since } k = \frac{2\pi}{\lambda}$$

$$x = 0, \frac{v}{2f}, \frac{2v}{2f}, \frac{3v}{2f}, \frac{4v}{2f}, \quad \text{since } \lambda = \frac{v}{f}$$



What is the velocity and accelerationen ?

Displacement:

$$y(x,t) = 2A \sin(kx) \sin(\omega t)$$

Wavefunction

Velocity:

$$v_y(x,t) = \frac{\partial y(x,t)}{\partial t} \rightarrow v_y(x,t) = 2A\omega \sin(kx) \cos(\omega t)$$

Acceleration:

$$a_y(x,t) = \frac{\partial v_y(x,t)}{\partial t} = \frac{\partial^2 y(x,t)}{\partial t^2} \rightarrow a_y(x,t) = -2A\omega^2 \sin(kx) \sin(\omega t)$$

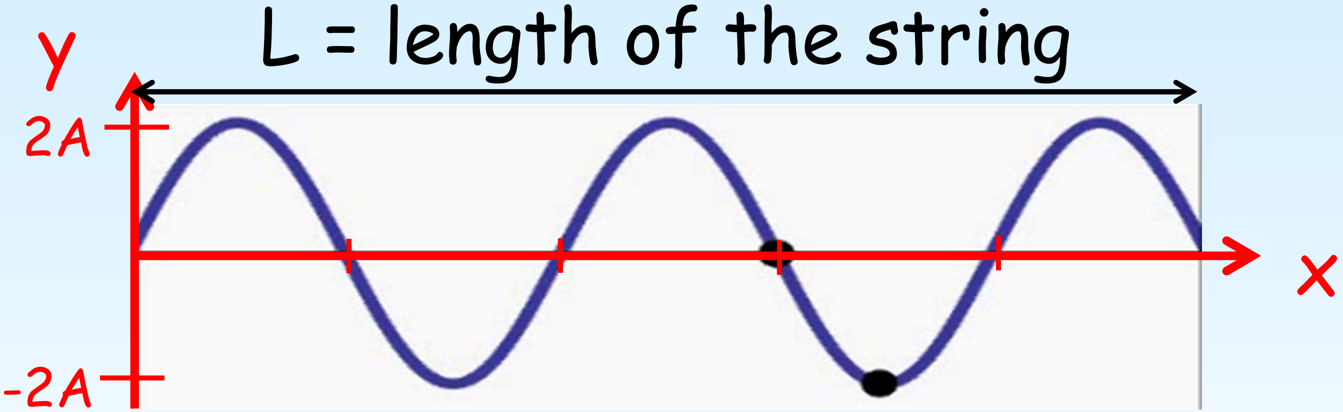
String instruments

Octobasse violin

https://www.youtube.com/watch?v=12X-i9YHzmE

Mechanical waves: String instruments

Example of a standing wave on a string:



Nodes: $0 \quad \frac{\lambda}{2} \quad \lambda \quad \frac{3\lambda}{2} \quad 2\lambda \quad \frac{5\lambda}{2} = L$

$$\lambda = \frac{2L}{5}$$

$$\boxed{\lambda = \frac{v}{f}}$$

$$f = \frac{5v}{2L}$$

The speed of the waves that build up the standing wave.

Mechanical waves: String instruments

Strings of length L having nodes at both ends:

Nodes when $\sin(kx) = 0$

$$x = 0, \frac{\pi}{k}, \frac{2\pi}{k}, \frac{3\pi}{k}, \ldots$$

$$= 0, \frac{\lambda}{2}, \frac{2\lambda}{2}, \frac{3\lambda}{2}, \ldots$$

$$L = n\frac{\lambda}{2} \quad (n = 1, 2, 3, \ldots)$$

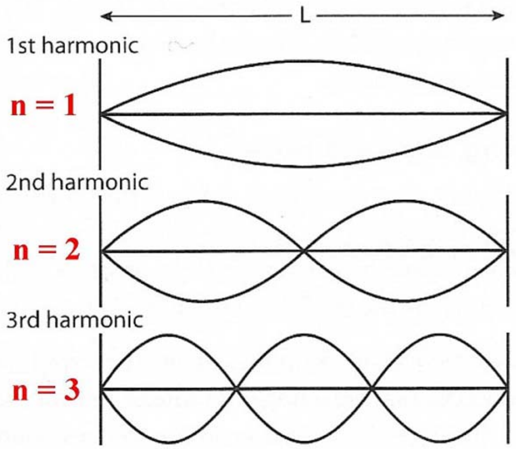
$$\lambda_n = \frac{2L}{n} \quad (n = 1, 2, 3, \ldots)$$

$$\lambda = 2L / n = v / f$$

$$f_n = n\frac{v}{2L} = nf_1 \quad (n = 1, 2, 3, \ldots)$$

f_1, f_2, f_3.... Harmonic frequencies
f_1: Fundamental frequency
f_2, f_3, f_4.... Overtones

$\lambda_n = \frac{2}{n}L$ $f_n = \frac{v}{\lambda_n}$ ← samma för alla n



En halv period

$\lambda_1 = \frac{2}{1}L$ $f_1 = \frac{v}{\lambda_1} = \frac{v}{2L}$

En hel period

$\lambda_2 = \frac{2}{2}L$ $f_2 = \frac{v}{\lambda_2} = \frac{v}{L} = 2f_1$

Tre halva perioder

$\lambda_3 = \frac{2}{3}L$ $f_3 = \frac{v}{\lambda_3} = \frac{v}{2/3L} = 3f_1$

Mechanical waves: String instruments

$$f_1 = v/2L$$

$$v = \sqrt{F/\mu}$$

$$f_1 = \frac{1}{2L}\sqrt{\frac{F}{\mu}}$$

Long string:	Low frequency
Thick string:	Low frequency
Large tension:	High frequency

SUMMARY

Mechanical waves

Mechanical waves: Summary

The sinusoidal oscillations on a string are described by the wave equation

$$\frac{\partial^2 y(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y(x,t)}{\partial t^2}$$

which has the wavefunction as a solution

$$y(x,t) = A\cos(kx-\omega t)$$

Velocity and acceleration are obtained by derivation

$$v_y(x,t) = \frac{\partial y(x,t)}{\partial t} = \omega A\sin(kx-\omega t)$$

$$a_y(x,t) = \frac{\partial^2 y(x,t)}{\partial t^2} = -\omega^2 A\cos(kx-\omega t) = -\omega^2 y(x,t)$$

Wave velocity

$$v = \lambda/T = \omega/k \qquad v = \sqrt{\frac{F}{\mu}}$$

Mechanical waves: Summary

Average power

$$P_{av} = \frac{1}{2}\mu(\omega A)^2 v = \frac{1}{2}\sqrt{\mu F}(\omega A)^2$$

The power function

$$P(x,t) = 2P_{av}\sin^2(kx-\omega t)$$

Wavefunction for a standing wave

$$y(x,t) = 2A\sin(kx)\sin(\omega t)$$

Fundamental frequency

$$f_1 = \frac{1}{2L}\sqrt{\frac{F}{\mu}}$$

$f_n = nf_1 \quad n = 2, 3, 4\ldots$